CHARACTERISATIONS OF SOME CLASSES OF FINITE SOLUBLE GROUPS

by

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Declaration

I declare that this thesis has been composed by myself and has not been accepted in any previous application for a degree. The work is my own, except where specifically acknowledged.

Summary

The main aim of this thesis is to characterise the so-called "ranked" saturated formations - that is, those saturated formations which are completely determined by the ranks of chief factors of the groups contained in them. This is motivated by recent work of Gaschütz, of Hawkes, and of Heineken. To achieve this characterisation, an area of Clifford theory is developed in Chapter I. In Section 3 of the first chapter, we introduce a method of applying a theorem of Clifford (see I.1.4) when arbitrary fields of non-zero characteristic are involved without resorting to the theory of representations over algebraic number fields. This method is developed by means of the technique (described in Section 2) of extending the base field of a module to a Galois extension of that field. In Section 4, some corollaries of Theorem I.3.6 are employed to prove a partial generalisation (Theorem I.4.2) of a well-known result due, independently, to Swan and Dade.

Chapter II is a brief resume of the elementary theory of finite soluble groups. In particular, in Section 4 the notion of a ranked saturated formation is formally defined, and the relationship between ranked saturated formations and the so-called Gaschütz classes is noted. Chapter III contains the characterisation theorems. Theorem III.2.3 gives arithmetical criteria for a saturated formation to be ranked, and provides a local definition of such formations. Theorem III.2.5 proves that the ranked saturated formations are precisely the Gaschütz classes which are formations, and that these are precisely the subgroup-closed Gaschütz classes.

The thesis finishes in Chapter IV with a full characterisation of the saturated formations which are complelctly determined by the absolute ranks of chief factors of the groups contained in them.

Conventions and notation

Whenever possible, the notation and conventions of Huppert [26] have been observed. Thus all groups are assumed finite and all modules are assumed finite dimensional. We list the notation not specifically defined in the text. Let A and G be groups, and let H be a subgroup of G. Let K and L be fields and let V be a K G-module. We take m and n to be positive integers with $m \leq n$ and π to be a set of prime numbers.

$H \leq G$	H is a subgroup of G
$H < G$	H is a proper or trivial subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$H \triangleleft G$	H is a proper or trivial normal subgroup of G
$H <\cdot\, G$; $H \cdot\!\leq G$	H is a maximal or minimal subgroup respectively of G
$H \triangleleft\cdot\, G$; $H \cdot\!\trianglelefteq G$	H is a maximal or minimal normal subgroup respectively of G
$Z(G)$	the centre of G
$F(G)$	the Fitting subgroup of G
$\Phi(G)$	the Frattini subgroup of G
$\mathrm{Soc}(G)$	the socle of G
π'	$\mathbb{P} \setminus \pi$
$O_{\pi}(G)$	the largest normal π-subgroup of G

$O_{\pi',\pi}(G)$	the normal subgroup of G with $\dfrac{O_{\pi',\pi}(G)}{O_{\pi}(G)} = O_{\pi'}(G / O_{\pi}(G))$.
$O^{\pi}(G)$	$\bigcap\{N \trianglelefteq G : G / N \text{ is a } \pi\text{-group}\}$
$O^{\pi',\pi}(G)$	$O^{\pi'}(O^{\pi}(G))$
$C_G(H)$	$\{x \in G : h^x = h \text{ for all } h \in H\}$
$N_G(H)$	$\{x \in G : H^x = H\}$
$A \wr G$	the regular wreath product of A with G (see [26], I, 15.1)
$A \wr_H G$	the restricted wreath product of A with G over H (see [26], I, 15.10)
$V \big\vert_H$	the restriction of V to H
Ker (G on V)	$\{g \in G : vg = v \text{ for all } v \in V\}$
J (K G)	the Jacobson radical of K G
K^+	the additive group of K
$K^\times$	the multiplicative group of K
$[V] \cdot G$	the semidirect product of V with G (see [26], I, 14.4)
L : K	L is a field extension of K
$\lvert L : K \rvert$	the degree of the extension of L : K
Gal (L : K)	the Galois group of the extension L : K
$\binom{n}{m}$	$\dfrac{n(n-1)\ldots(n-m+1)}{m(m-1)\ldots 1}$
$m \mid n$	m divides n

$m \nmid n$	m does not divide n
Z_n	the cyclic group of order n
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{P}$	the set of prime numbers

Introduction

Introduction

After Galois' theory of the solution of polynomial equations by radicals, the biggest impetus to the study of finite, soluble groups has been given by Hall's generalisation of Sylow Theory (see [20]), and the consequent characterisation of finite p-soluble groups. This theory, together with Carter's [2] discovery of a characteristic conjugacy classes of self-normalising nilpotent subgroups in every finite soluble group, led Gaschütz [15] to introduce the notions of formations and saturated formations. Subsequently, these ideas were developed into the theories of Schunck classes and, from a conceptually dual approach, of Fitting classes (the reader is referred to Gaschütz [19] for more information about the terms and theories which we mention without other reference in this introduction). In the process of this abstractification, the arithmetical beginnings of the theories have lost their initial importance. Recently, however, there has been a resurgence of interest in this arithmetical approach, with papers from Gaschütz ([16], [17] and [18]) and Hawkes [24] concerned with subgroups which are determined in a finite soluble group via a set of powers of prime numbers, rather than just the primes themselves, as Hall subgroups are.

The work of Hawkes [24] generalises that of Gaschütz [16], [18]. If Ω is a set of prime powers, then a Gaschütz Ω-subgroup H of a group G has the following properties:

Property 1: for all $U \lessdot H$, the index $|H : U|$ is in Ω, and

Property 2: whenever $H \leq R \lessdot S \leq G$, the index $|S : R|$ is in $\mathbb{N} \setminus \Omega$

Hawkes shows that if G is a finite soluble group then a Gaschütz Ω-subgroup H of G, if it exists, may be found as a projector for a certain Schunck class $P\mathfrak{F}_{\Omega}$. These classes are known as Gaschütz classes. One of our aims is to find sets Ω of prime powers for which the Gaschütz class $P\mathfrak{F}_{\Omega}$ is a saturated formation (Theorems III.2.6 and III.2.3). This we do by considering those saturated formations whose elements are completely described by the ranks of their chief factors. In doing so, we generalise work of Heineken [25]. It is evident that we shall need an adequate description of the dimension of an irreducible module of a finite soluble group over an arbitrary finite field in terms of the prime numbers which divide the order of that group. This description is the aim of Chapter I.

Section 1 of the first chapter is designed to put into context some of the problems encountered with direct applications of Clifford's theory. The opportunity is also taken to record some elementary and well-known facts.

The technique of extending the base field of a module V for a given group G to a splitting field for G has been well documented (see, for instance, Fein [11], [12], Isaacs [27], Chapter 9, and various chapters in Huppert [26] and Curtis and Reiner [7]). Similar results are also available for extending the base field of V to arbitrary field extensions, and some of these are proved in Section2 of Chapter I. For convenience, attention is restricted to Galois extensions of the base field, for reasons that will be clear when this section is examined. Although this theory is not original - the author first encountered it in some lecture notes prepared by Professor B. Huppert at

The University of Mainz, West Germany - the material is not accessible in the available literature. Our applications are, however, new. For instance, in Section 3 we examine the restriction to a normal subgroup N of a finite group G of an irreducible G-module V over a field of non-zero characteristic. In particular, we are interested in the case when $V\big|_N$ is homogeneous. In order to apply Clifford's theory in this event (in particular, to apply I.1.4) we require the base field of V to be sufficiently large for the irreducible submodules of $V\big|_N$ to be absolutely irreducible. In general, of course this is far from being the case, and we have to employ the techniques introduced in Section 2. However, over a larger field, the resultant irreducible G-modules need not be homogeneous on restriction to N (see example I.3.4). Theorem I.3.6 is designed to make sense of this situation and to help us around this difficulty by looking at a particular normal subgroup of G containing N. The section is completed by stating some corollaries and lemmas which are useful in inductive situations, and in particular describe $V\big|_N$ (with the above notation) when $V\big|_N$ is homogeneous and N is a maximal normal subgroup of G. These results become essential knowledge in order to prove the main theorem in Section 4. This last section of Chapter I deals with generalising the celebrated Swan - Dade Theorem ([31], Theorem 6, and [8]) to finite soluble groups and irreducible modules over arbitrary fields of non-zero characteristic. The major result (Theorem I.4.2) does just this and, happily, the Swan - Dade Theorem in our special case is available as an easy corollary. The results of Section 3 and of the first half of Section 4 are used to investigate the dimensions of irreducible modules of certain groups - for instance, of p-groups. Although this does not actually provide any new information, we are able to demonstrate how Theorem I.3.6 can be used to avoid the more traditional techniques of examining modules over algebraic number fields, and employing constant reduction - a manoeuvre of limited applicability at the best of times. The chapter closes with some results of Fein and of Förster concerning representations of direct products of groups.

We use Chapter II to introduce some of the more fundamental finite group theory which will be required. Sections 1, 2 and 3 consist of results available in the literature gathered together in a convenient form for later use. In Section 4 we describe how to form formations of groups via so-called ranking functions. In particular, these formations are completely determined by the ranks of the chief factors of the groups contained in them. We call them ranked formations. We see that such formations which are also saturated are precisely the Gaschütz classes $P\mathfrak{F}_{n}$ mentioned earlier which are also formations. The ranking functions corresponding to the ranked saturated formations are closely examined in the opening section of Chapter III. After showing that the ranked saturated formations are subnormal subgroup-closed - a basically technical result of crucial importance and involving a particularly complicated proof - we discover that these ranking functions have seven properties. In Section 2 we show that in fact a ranking function having these seven properties determines a saturated formation, and that therefore (III.2.3) they provide a set of necessary and sufficient conditions for a ranking function to determine a saturated formation. One corollary of this Theorem

(III.2.5) proves that the properties of a ranking function associated with a saturated formation of full characteristic listed by Heineken [25] are also sufficient. The second main theorem of Section 2 in the third chapter (III.2.6) recalls that the set of Gaschütz classes which are formations is precisely the set of ranked saturated formations, and that such Schunck classes are precisely those Gaschütz classes which are subgroup closed. Unfortunately, the ranking functions corresponding to the ranked saturated formations are sufficiently complicated as to defy an easy description, although a group theoretic problem has now been reduced to a totally number theoretic problem. Section 3 of Chapter III is concerned with examples and conjectures which attempt to give some idea of what these ranking functions defining saturated formations may look like. The difficulties in describing these saturated formations is in great contrast to the saturated formations investigated in Section IV. Here we characterise the saturated formations which are completely described by the absolute ranks of chief factors of the groups contained in them. Theorem IV.13 and IV.14 provide a complete and more immediate description of such saturated formations than that available for the saturated formations of Chapter III.

Chapter I

Chapter I. Representation Theory

1. Some preliminary results and observations

In the theory of finite soluble groups representation theory plays a key role , and the normal structure of a finite soluble group makes Clifford's theory of the representations of normal subgroups stand out as playing the most important part of all. The aim of this chapter is to closely examine and to develop some aspects of this theory in order to establish a particularly interesting factorisation of the degree of an irreducible representation of a finite soluble group. This factorisation will be crucial in the sequel. In this section we remind the reader of some elementary representation theory, beginning with the basic result in Clifford's theory.

I.1.1 Definition (a) Let N be a normal subgroup of the group G and let D be a representation of N. For each $g \in G$ the map D^g defined by

$$D^g(n) = D(n^g)$$

for all $n \in N$ is again a representation of N which is G-conjugate to D.

(b) Let G be a group, let K be a field and let V be a KG-module. If V is a direct sum of pairwise-isomorphic irreducible KG-modules then we say that V is a homogeneous KG-module.

I.1.2 Theorem (Clifford [5]) Let G be a group with normal subgroup N and let V be an irreducible G-module over some field K.

(a) If W is an irreducible KN-submodule of V, then $V = \sum_{g \in G} Wg$. Each Wg is an irreducible KN-module and hence V is a completely reducible KN-module.

(b) The homogeneous components, V_i, of the KN-module V are defined in the following way: Let $W_1, \dots, W_n$ be representatives of the isomorphism types of irreducible KN-submodules of V. Set

$$V_i = \sum_{\substack{W \leq V \\ W \cong W_i}} W \quad (i = 1, \dots, n).$$

Then V_i is a direct sum of some KN-submodules of V which are isomorphic to W_i. We have

$$V = \bigoplus_{i=1}^{n} V_i .$$

(c) Let F_i be the irreducible representation of N on W_i. Then $W_i g$ affords the representation $F_i^{g^{-1}}$ of N.

(d) The homogeneous components V_i of the KN-module V are transitively permuted by right multiplication by elements of G.

(e) For $j \in \{1, \dots, n\}$ we have

$$\{g \in G : V_j g = V_j\} = \{g \in G : F_j^g \text{ is equivalent to } F_j\}.$$

Denoting this group by U_j we have that V_j is an irreducible KU_j-module and

$$V \cong V_j \otimes_{KU_j} KG = V_j^G .$$

(f) Let D be the representation of G on V. The irreducible parts of $D|_N$ are precisely the G-conjugates F^g of an irreducible representation F of N and each appear with the same multiplicity e.

(g) If D affords the character χ and F affords the character φ, then

$$\chi|_N = e \sum_{i=1}^{n} \varphi^{g_i}$$

where $\{g_i : i=1, \dots, n\}$ is a transversal of the subgroup

$$U = \{g \in G : F^g \text{ is equivalent to } F\}$$

and $\varphi^g(n) = \varphi(n^g)$ for $n \in N$ and $g \in G$.

It is hard to emphasise enough the importance of this theorem. We shall use the information contained within its statement frequently, referring to it simply as "Clifford's Theorem" and, often, without elaborating on its application to a specific situation. The fame of the theorem is such that this should present no obstacle to comprehension.

Nonetheless, there are situations in which Clifford's theorem fails to provide us with any useful information, namely, with the notation of I.1.2, when $V|_N$ is homogeneous. Clifford himself went part of the way to filling this gap by employing the concept of a projective representation.

I.1.3 Definition Let G be a group and K a field. A projective K-representation of G is a map $P : G \rightarrow GL(n,K)$ such that for every $g,h \in G$, there exists a scalar $\alpha(g,h) \in K$ such that

$$P(g)\, P(h) = \alpha(g,h)\, P(gh).$$

The integer n is the degree of P. If V is the natural module for $GL(n,K)$ then P is said to be irreducible if $P(G)$ acts as a set of K-linear transformations on no proper K-subspace of V.

I.1.4 Theorem (Clifford [5]) Let G be a group with a normal subgroup N and let K be an arbitrary field. Let V be an irreducible KG-module affording the representation D and suppose that $V|_N$ is a homogeneous direct sum of s absolutely irreducible KN-modules. Then there exist irreducible projective representations P_1 and P_2 of G such that

(i) $D(g) = P_2(g) \otimes P_1(g)$ for $g \in G$;

(ii) the restriction of P_2 to N is the ordinary representation, R say, afforded by an irreducible KN-submodule of $V|_N$ and $\deg P_2 = \deg R$; and (iii) P_1 has degree s and has N in its kernel; that is, P_1 maps

N onto the identity element of $GL(s,K)$ and may therefore be considered as as irreducible projective representation of G/N.

Proof Although the statement given here is somewhat stronger than that given by Huppert ([26], V, 17.5), it is clear that his proof goes through with only minor alterations.
q.e.d.

As we suggested above, this theorem still does not furnish as much information as we would like, the condition that the irreducible submodules of $V|_N$ should be absolutely irreducible proving to be too restrictive in practice. The notion of absolute irreducibility will be formally defined in section 2 of this chapter, where we examine the consequences of enlarging the base field of a module. In section 3 we use the knowledge gained in section 2 to develop a technique which will often help us over the difficulties of applying I.1. 4, without resorting to "traditional" methods involving constant reduction and Schur's theory of projective representations.
Theorem I.1.4 will be used in this thesis in inductive argments with G/N cyclic of prime order. The degrees of irreducible projective representations of such groups over certain fields are easily calculated, as we shall see in Lemma I.1.8, with the help of some preliminary lemmas. Lemma I.1.6 will actually contain more information than is required in this section, but it is convenient at this stage to make all the number-theoretic statements which will be needed in the course of the thesis. Lemma I.1.7 will also be frequently used throughout this work and gives a foretaste of the main result of section 4 in Chapter I.

I.1.5 Notation and Definition Let m and n be positive integers.
(a) We denote by (m,n) the highest common factor (h.c.f.) of m and

n and by $[m,n]$ the lowest common multiple (l.c.m.) of m and n.

(b) If $(m,n) = 1$ and $m \neq 1 \neq n$ then the <u>degree of m (mod n)</u>, written $\deg_n m$, is the smallest positive integer d such that $n \mid m^d - 1$.

(c) We write $n^k \parallel m$ to indicate that k is the largest positive integer such that $n^k \mid m$.

<u>I.1.6 Lemma</u> Let a, b and c be positive integers such that $(a,b) = (a,c) = 1$ and $1 \notin \{a,b,c\}$, and let p be a prime such that $p \nmid a$.

(a) (Fermat) $p \mid a^{p-1} - 1$.

(b) If n is a positive integer such that $b \mid a^n - 1$ then $\deg_b a \mid n$.

(c) Let $d = \deg_p a$ and $p^k \parallel a^d - 1$. Let m be a positive integer.

(i) If p is odd then $\deg_{p^m} a = \begin{cases} d & \text{if } m \leqslant k \\ dp^{m-k} & \text{if } m \geqslant k \end{cases}$

(ii) If $p = 2$ then $d = 1$. If $2^j \parallel a + 1$ then

$$\deg_{2^m} a = \begin{cases} 1 & \text{if } m \leqslant k \\ 2 & \text{if } k < m \leqslant j + k \\ 2^{m-(k+j)+1} & \text{if } j + k \leqslant m \end{cases}$$

(d) $\deg_{[b,c]} a = [\deg_b a, \deg_c a]$.

(e) Let $e = \deg_c a$. Then

$$e . \deg_b a^e = \deg_{[b,c]} a.$$

<u>Proof (a)</u> [22], 6.1, Theorem 71.

(b) [22], 6.8, Theorem 88.

(c) By induction on m. If $m \leqslant k$, then the statement is clearly true. Suppose then that $m > k$ and that the statement holds for all integers less than or equal to $m - 1$.

<u>Case 1: p is odd.</u> By the induction hypothesis there exists an

integer w such that

$$a^{dp^{m-1-k}} = p^{m-1} w + 1 .$$

Using the binomial theorem we obtain the following sequence of equalities:

$$\begin{aligned}
a^{dp^{m-k}} &= (a^{dp^{m-1-k}})^p \\
&= (p^{m-1} w + 1)^p \\
&= (p^{(m-1)p} w^p + \binom{p}{1} w^{p-1} p^{(m-1)(p-1)} + \ldots + \binom{p}{p-2} w^2 p^{2(m-1)}) \\
&\qquad + \binom{p}{p-1} wp^{m-1} + 1 \\
&= p^m(p^{(m-1)p-m} w^p + \binom{p}{1} w^{p-1} p^{(m-1)(p-1)-m} + \ldots + \binom{p}{p-2} w^2 p^{m-2}) \\
&\qquad + wp^m + 1 \\
&= p^m w_o + 1,
\end{aligned}$$

where $w_o = p^{(m-1)p-m} w^p + \binom{p}{1} w^{p-1} p^{(p-1)(m-1)-m} + \ldots + \binom{p}{p-2} w^2 p^{m-2} + w.$

Thus it follows that

$$p^m \mid a^{dp^{m-k}} - 1,$$

and by (b) we have

(1) $\deg_{p^m} a \mid dp^{m-k}.$

It also a consequence of (b) that

(2) $d = \deg_p a \mid \deg_{p^m} a.$

Suppose that

(3) $p^m \mid a^{dp^{m-k-1}} - 1.$

Since $a^{dp^{m-k-1}} - 1 = p^{m-1} w$, it follows that $p \mid w$.

Notice that if (3) holds then $m \geqslant k + 2$. For if $m = k + 1$, then we should have $p^m \mid a^d - 1$, contrary to the definition of k. Let x denote the integer such that

(4) $a^{dp^{m-k-2}} - 1 = p^{m-2} x.$

integer w such that

$$a^{dp^{m-1-k}} = p^{m-1} w + 1 .$$

Using the binomial theorem we obtain the following sequence of equalities:

$$\begin{aligned}
a^{dp^{m-k}} &= (a^{dp^{m-1-k}})^{p} \\
&= (p^{m-1} w + 1)^{p} \\
&= (p^{(m-1)p} w^{p} + \binom{p}{1} w^{p-1} p^{(m-1)(p-1)} + \dots + \binom{p}{p-2} w^{2} p^{2(m-1)}) \\
&\qquad + \binom{p}{p-1} wp^{m-1} + 1 \\
&= p^{m}(p^{(m-1)p-m} w^{p} + \binom{p}{1} w^{p-1} p^{(m-1)(p-1)-m} + \dots + \binom{p}{p-2} w^{2} p^{m-2}) \\
&\qquad + wp^{m} + 1 \\
&= p^{m} w_{0} + 1,
\end{aligned}$$

where $w_{0} = p^{(m-1)p-m} w^{p} + \binom{p}{1} w^{p-1} p^{(p-1)(m-1)-m} + \dots + \binom{p}{p-2} w^{2} p^{m-2} + w.$

Thus it follows that

$$p^{m} \mid a^{dp^{m-k}} - 1,$$

and by (b) we have

(1) $\deg_{p^{m}} a \mid dp^{m-k}.$

It also a consequence of (b) that

(2) $d = \deg_{p} a \mid \deg_{p^{m}} a.$

Suppose that

(3) $p^{m} \mid a^{dp^{m-k-1}} - 1.$

Since $a^{dp^{m-k-1}} - 1 = p^{m-1} w$, it follows that $p \mid w$.

Notice that if (3) holds then $m \geqslant k + 2$. For if $m = k + 1$, then we should have $p^{m} \mid a^{d} - 1$, contrary to the definition of k. Let x denote the integer such that

(4) $a^{dp^{m-k-2}} - 1 = p^{m-2} x.$

By expanding $(p^{m-2}x+1)^p$ as above, we obtain

$$w = p^{(m-2)p-(m-1)}x^p + \binom{p}{1}x^{p-1}p^{(p-1)(m-2)-(m-1)} + \ldots + \binom{p}{p-2}x^2p^{m-3} + x$$

$$= p^{(m-2)p-(m-1)}x^p + \binom{p}{1}x^{p-1}p^{(p-1)(m-2)-(m-1)} + \ldots + p\binom{p-1}{2}x^2p^{m-3} + x.$$

Since p is odd, $m \geqslant 3$ and $p \mid w$, it must follow from this identity that $p \mid x$ also. But (4) would then imply that $p^{m-1} \mid a^{dp^{m-k-2}} - 1$, contrary to the induction hypothesis that $\deg_{p^{m-1}} a = dp^{m-k-1}$. Hence (3) must fail, and this fact taken together with (1) and (2) yields

$$\deg_{p^m} a = dp^{m-k},$$

as desired to complete the proof in this case.

<u>Case 2: $p = 2$</u>. Since by hypothesis $2 \nmid a$, we have $2 \mid a-1$ and, consequently, $d = 1$. Notice that $2^j \parallel a+1$, whence

$$2^{k+j} \parallel (a-1)(a+1) = a^2 - 1.$$

Since by (b) we then have $\deg_{2^{k+j}} a \mid 2$, and because $2^k \parallel a-1$, we must have

$$\deg_{2^{k+i}} a = 2 \text{ for } 1 \leqslant i \leqslant j.$$

Therefore we may assume that $m \geqslant k+j+1$. The argument now runs just as in case 1.

(d) and (e) are straightforward to prove.

q.e.d.

<u>I.1.7. Lemma</u> (Huppert [26], II, 3, 10) Let A be an abelian group which is faithfully and irreducibly represented on a module V of dimension n over the field $GF(p^f)$. Then A is cyclic and n is the smallest positive integer such that $|A| \mid p^{nf} - 1$ – i.e., $\deg_{|A|} p^f = n$.

<u>I.1.8 Lemma</u> Let K be a field.

(a) If G is a cyclic group of order q, a prime, and $K = GF(p^f)$

then the irreducible projective K - representations of G have degree 1, q or $\deg_q p^f$.

(b) If G is a cyclic group and K is algebraically closed then the degree of any irreducible projective K-representation of G is 1.

Proof Let G be a cyclic group and let P be an irreducible projective K-representation of G with degree n. If π denotes the natural homomorphism from $GL(n,K)$ onto $PGL(n,K) = GL(n,K) / Z(GL(n,K))$, then it is a well-known (and easily proved) fact that the map

$$\pi o P : G \to PGL(n,K)$$

is a homomorphism. Let $Z = Z(GL(n,K))$. We see that $\pi(P(G)) = ZP(G) / Z$ is a subgroup of $PGL(n,K)$ and so $ZP(G)$ is a centre-by-cyclic, and hence abelian, subgroup of $GL(n,K)$. Let V be the natural n-dimensional K-module for $GL(n,K)$. Since P is an irreducible projective K-representation it follows at once that $ZP(G)$ acts irreducibly on V. Indeed, as $ZP(G) \leq GL(n,K)$ this action must even be faithful. Notice that we must therefore have $q \neq \mathrm{char}\, K$.

(a) We have $G \cong Z_q$, where q is a prime, and $K = GF(p^f)$. By I.1.7 then $ZP(G)$ is cyclic. Also, $ZP(G) / Z$ is a homomorphic image of the simple group G and so either $P(G) \leq Z$ or else $|Z\,P(G)| = q(p^f - 1)$. If $P(G) \leq Z$ then the irreducibility of V as a Z-module forces $n = 1$. We may therefore restrict our attention to the case $|ZP(G)| = q(p^f - 1)$.

Case 1: $q \nmid p^f - 1$. Then by I.1.6 (d) and I.1.7 we have

$$n = \deg_{q(p^f-1)} p^f = [\deg_q p^f, \deg_{(p^f-1)} p^f] = \deg_q p^f.$$

Case 2: $q \mid p^f - 1$. Let $q^k \,\|\, p^f - 1$ and let $p^f - 1 = q^k m$.

By I.1.6 (d) and I.1.7 again we have

$$n = \deg_{q(p^f-1)} p^f = \deg_{q^{k+1}m} p^f = [\deg_{q^{k+1}} p^f, \deg_m p^f] = \deg_{q^{k+1}} p^f$$

and by I.1.6 (c) we therefore have

$$n = q.\ \deg_q p^f = q.$$

(b) If K is algebraically closed then it is immediate that $n = 1$.

q.e.d.

Our next lemma also uses parts of I.1.6.

I.1.9 Lemma let p be a prime and let a and b be positive integers with $1 \leqslant b \leqslant a$. Then there exist non-abelian metacyclic p-groups P and $\bar{P}$ such that

(i) $\bar{P}$ is a quotient of P, and

(ii) there are self-centralising cyclic normal subgroups N and $\bar{N}$ of P and $\bar{P}$ respectively such that $P / N \cong Z_{p^a}$, $\bar{P} / \bar{N} \cong Z_{p^b}$, $N \cong Z_{p^{a+1}}$ and $\bar{N} \cong Z_{p^{b+1}}$ if p is odd, and $N \cong Z_{2^{a+2}}$ and $\bar{N} \cong Z_{2^{b+2}}$ if $p = 2$.

Proof We must consider two separate cases.

Case 1: p is odd Let N be a cyclic group of order p^{a+1}. By [26], I, 13.19 (b) a Sylow p-subgroup A of Aut(N) has order p^a and is cyclic. Let $P = [N]A$. The group P and normal subgroup N clearly have the desired properties.

Let y be a generator of N. Then we may take as a generator of A the automorphism α of N with

$$y^{\alpha} = y^{p+1}.$$

For by I.1.6 (c)(i) we have

(1) $p^n \mid (1+p)^{p^{n-1}} - 1$ for $n \geqslant 1$

and

(2) $\deg_{p^n} (1+p) = p^{n-1}$ for $n \geqslant 1$.

In particular, it follows from (2) that α has order p^a. In proving the lemma we may obviously assume that $b < a$. Let $A^* = \langle \alpha^{p^b} \rangle$ and $N^* = \langle y^{p^{b+1}} \rangle$. Then $A_0 = A / A^*$ is cyclic of order p^b and $N_0 = N / N^*$ is cyclic of order p^{b+1}. We claim that $N^* A^* \trianglelefteq P$. Clearly $N^* \trianglelefteq P$, for since N is cyclic we have

$$N^* \text{ char } N \trianglelefteq P.$$

Observing the identity

$$(\alpha^y)^j = \alpha^j \, y^{-(p+1)^j + 1}$$

holds in P for $1 \leq j \leq p^a$ we obtain

(3) $(\alpha^{p^b})^y = (\alpha^y)^{p^b} = \alpha^{p^b} \, y^{-(p+1)^{p^b} + 1}$.

From (1) we have $p^{b+1} \mid (p+1)^{p^b} - 1$, and so

(4) $y^{-(p+1)^{p^b} + 1} = (y^{(p+1)^{p^b} - 1})^{-1} = y^{xp^{b+1}} \in N^*$ for some $x \in \mathbb{Z}$.

Hence by (3), $A^{*y} \leq A^* N^* = N^* A^*$, and this justifies our claim that $N^* A^* \trianglelefteq P$.

Set $\bar{P} = P / N^* A^*$. Since

$$N_o = N / N^* \cong N A^* / N^* A^*$$

and

$$A_o = A / A^* \cong N A / N A^* = P / N A^*$$
$$\cong \bar{P} / (N A^* / N^* A^*),$$

we see that $\bar{P}$ is metacyclic of order p^{2b+1} with normal subgroup $\bar{N} = \langle y N^* A^* \rangle$ of order p^{b+1} and subgroup $\bar{A} = \langle \alpha N^* A^* \rangle$ of order p^{b+1} such that $\bar{N} \bar{A} = \bar{P}$ and $\bar{N} \cap \bar{A} = 1$. Since

$$(y N^* A^*)^{\alpha N^* A^*} = y^{\alpha} N^* A^* = y^{p+1} N^* A^*,$$

it is clear that $\bar{A}$ may be identified with a Sylow p-subgroup of Aut $(\bar{N})$ and the lemma goes through in this case.

<u>Case 2: p = 2.</u> If $a = 1$ then we observe that the dihedral group of order 8 has the properties we require of P and the lemma clearly holds. We may therefore assume that $a \geq 2$. Let N be a cyclic group of order 2^{a+2}. By [26], I, 13.19 (c) there is a cyclic 2-subgroup A of Aut(N) of order 2^a. Set $P = [N] \cdot A$. The proof now goes through in a manner analogous to that of case 1.

q.e.d.

To conclude this introductory section to Chapter I we list some elementary results concerning representations which will be used without comment in the work which follows.

<u>I.1.10 Theorem</u> Let G be a group and let K be a field.

(a) $J(KG) = \{a \in KG : Va = 0 \text{ for all irreducible } KG\text{-modules } V\}$

(b) A KG-module W is completely reducible if and only if $W\,J(KG) = 0$.

(c) If V is a faithful, irreducible G-module over K and $1 \neq N \trianglelefteq G$, then

$\{v \in V : vn = v \text{ for all } n \in N\} = 0.$

(d) If V is as in (c) and char $K = p \neq 0$ then

$$O_p(G) = 1.$$

<u>Proof (a)</u> This is an alternative definition of the Jacobson Radical of KG (see [26], V, section 2).

(b) Certainly, if W is completely reducible, then $W\,J(KG) = 0$ by (a). Suppose that $W\,J(KG) = 0$. Then W may be considered as a module for the K-algebra $KG / J(KG)$, which by [7], 24.4 is semisimple. By [7], 25.8, the module W is therefore a completely reducible KG-module.

(c) Let $U = \{v \in V : vn = v \text{ for all } n \in N\}$ and suppose that $U \neq 0$. We claim that U is a KG-module. Clearly U is a K-space. If $u \in U$ $g \in G$ and $n \in N$, then we have

$$ugn = un^{g^{-1}}g = ug.$$

Hence $ug \in U$ and U is a KG-module as claimed. However, V is an irreducible KG-module, and so we conclude that $V = U$. This contradicts the faithfulness of V for G, and so our initial supposition must have been false.

(d) [26], V, 5.17.

q.e.d.

Necessary and sufficient conditions for the existence of a faithful and irreducible module for a group over a field of characteristic $p \neq 0$ are well-known. The following criterion, though, is sufficient for our needs.

I.1.11 Lemma (Doerk [9] 1.3). Let G be a group with $O_p(G) = 1$ and such that G has at most two different minimal normal subgroups. Then G has a faithful, irreducible representation over a field K of characteristic $p \neq 0$.

2. Extending the base field of a module

Much of what follows in this section is well-known and , where possible, the reader will be referred to the relevant literature for proofs. Unfortunately, many authors are concerned only with extending the base field of a module to a splitting or algebraically closed field, whilst we require more general information than is accessible in the literature. The author's attention was drawn to these more general results by some notes prepared by Professor Huppert for an Arbeitsgemeinschaft at the University of Mainz in the Winter semester of 1979. It is Professor Huppert's approach to this subject which is used here.

Throughout this section L will denote a Galois extension of the field K, by which we understand that the extension L : K is finite, normal and separable (the reader is referred to [32], chapters 6 and 8 for any Field or Galois theoretic definitions and results which are assumed here). The Galois group associated with the extension L : K will be denoted by $\mathcal{H}$ and G will always be a group.

I.2.1 Definition Let V be a KG-module and let F be an arbitrary field extension of K. Since $F \otimes_K KG \cong FG$ (see [26], V,11 .3 (a)), the $F \otimes_K KG$-module $F \otimes_K V$ may be considered as an FG-module and as such is denoted by V^F. Clearly $\dim_K V = \dim_F V^F$.

Conversely, an FG-module W may be thought of as a KG-module and as such is denoted by W_K.

I 2.2 Lemma $J(L \otimes_K KG) = L \otimes_K J(KG)$. In particular, $J(LG) = L\, J(KG)$.

Proof [7], 69.10.

q.e.d.

An almost immediate consequence of I.2.2 is:

I.2.3 Lemma (a) A KG-module V is completely reducible if and only if V^L is a completely reducible LG-module.

(b) An LG-module W is completely reducible if and only if W_K is completely reducible.

Proof (a) By I.2.2 we have

$$V^L (J(LG)) = V^L\, L\, J(KG) = V^L J(KG) = (VJ(KG))^L.$$

Thus $V^L J(LG) = 0$ precisely when $V J(KG) = 0$.

(b) By I.2.2 again, we obtain

$$W\, J(LG) = W\, L\, J(KG) = W\, J(KG)$$

and so $W J(LG) = 0$ precisely when $W J(KG) = 0$.

Obviously $W J(KG) = W_K J(KG)$.

q.e.d.

A closer examination of the situation in I.2.3(b) yields the following result.

I.2.4 Theorem Let W be an irreducible LG-module.

(a) $\dim_K W_K = |L : K| \dim_L W$.

(b) The irreducible direct summands of W_K are all isomorphic to a particular irreducible KG-module, V say, and W is isomorphic to a direct summand of V^L.

Proof (a) [27], 9.18 (a).

(b) The KG-module W_K is completely reducible by I.2.3(b). The result now follows by [27], 9.18 (b).

The module V in I.2.4 is precisely determined (up to isomorphism) as we see in the corollary to the next theorem.

I.2.5 Theorem (Noether, Deuring). Let V_1 and V_2 be irreducible KG-modules. If $(V_1)^L$ and $(V_2)^L$ have a direct summand in common, then V_1 and V_2 are KG-isomorphic.

Proof [27], 9.7.

q.e.d.

I.2.6 Corollary If W is an irreducible LG-module then there is a uniquely determined (up to isomorphism) irreducible KG-module V such that W is isomorphic to direct summand of V^L.

Proof Such V exists by I.2.4. (b) and is uniquely determined by I.2.5.

A more complete description of W_K (in the notation of I.2.6) must be postponed until we have a satisfactory description of V^L when V is an irreducible KG-module. The key to providing the latter is held in part (a) of definition I.2.7.

I.2.7 Definition Let F be an arbitrary field extension of K and let W be an FG-module with F-character χ.

(a) Given $\sigma \in \mathrm{Gal}(F : K)$ we define an FG-module W^σ in the following way:

Let W^σ be an abelian group isomorphic to W with group isomorphism $i_\sigma : W \to W^\sigma$. We make W^σ into an F-vector space via

$$l\,(wi_{\sigma}) = ((l)\sigma^{-1}\, w)i_{\sigma}\,.$$

for $w \in W$ and $l \in F$, and W^{σ} then becomes an FG-module with G-action defined by

$$(wi_{\sigma})\, g = (wg\,)\, i_{\sigma}$$

for $w \in W$ and $g \in G$.

We call W^{σ} a <u>Galois conjugate of W over K.</u>

Let $\{w_1,\ldots,w_n\}$ be an F-basis of W and let

$$w_i\, g = \sum_{j=1}^{n} a_{ij}\,(g)\; w_j$$

for $i = 1,\ldots,n$, $g \in G$ and $a_{ij}\,(g) \in F$. It is easy to check that $\{w_1\, i_{\sigma},\ldots,\, w_n\, i_{\sigma}\}$ is an F-basis of W^{σ} and that

$$g \longmapsto ((a_{ij}\,(g))\,\sigma)$$

is a matrix representation of G on W^{σ}. In particular, χ^{σ} defined by

$$\chi^{\sigma}(g) = (\chi(g))\,\sigma$$

for all $g \in G$ is the F-character of W^{σ}.

(b) The extension $K\,(\chi(g) : g \in G)$ of K is denoted by $K\,(\chi)$.

<u>Note</u> : The definition of Galois conjugate given here does not precisely agree with that of Huppert ([26], V, §13). However, by replacing a field automorphism by its inverse where necessary, the results stated by Huppert in [26] are still applicable in this context. We now wish to show that we can also use the machinery set up by Isaacs in Chapter 9 of his book [27]. It is to this end that we state the next two results, the first of which is reached using Isaacs' definition of Galois conjugacy and is of great interest in its own right.

<u>I.2.8 Theorem</u> Let F be any field. Then the characters of non-equivalent irreducible F-representations of G are non-zero, distinct, and linearly independent.

<u>Proof</u> [27], 9.22.

q.e.d.

<u>I.2.9 Lemma</u> Let W be an LG-module with L-character χ.

(a) K (χ) is a Galois extension of K with abelian Galois group.

(b) Suppose that V is another LG-module with L-character φ.

Then V and W are Galois conjugate over K if and only if K (χ) = K (φ) and there is an automorphism $\rho \in$ Gal (K (χ) : K) such that $\chi(g)\rho = \varphi(g)$ for all $g \in G$.

<u>Proof (a)</u> This follows from the fact that K (χ) is contained in a splitting field for the polynomial $x^{|G|}-1$ over K.

(b) Suppose firstly that $V \cong W^{\sigma}$ for some $\sigma \in \mathcal{H}$, so that $\varphi(g) = \chi(g)\sigma$ for all $g \in G$. By (a), the intermediate field K (χ) of L : K is a normal extension of K and so (K $(\chi))\sigma$ = K (χ). Setting $\rho = \sigma|_{K(\chi)}$ we get the required result in this direction.

Now suppose that K (χ) = K (φ) and ρ is given in Gal (K (χ) : K) such that $\chi(g)\rho = \varphi(g)$ for all $g \in G$. Since L : K is a finite normal extension there is an automorphism $\sigma \in \mathcal{H}$ such that $\sigma|_{K(\chi)} = \rho$. It is clear that W^{σ} has character φ and it therefore follows from I.2.8 that $W^{\sigma} \cong V$.

q.e.d.

We have shown in Lemma I.2.9. (b) that our definition of Galois conjugacy is consistent with that of Isaacs under our extra hypothesis that L : K is Galois, and we may therefore quote from Isaacs [27] as well as from Huppert [26]. A series of results which build on I.2.9 help to improve our picture of V^L.

<u>I.2.10 Lemma</u> Let V be an irreducible KG-module. Then

(a) V^L is a direct sum of irreducible LG-modules which are pairwise Galois conjugate over K. If W is an irreducible LG-submodule of V^L, then every Galois conjugate of W over K appears as as a direct summand of V^L.

(b) Let $C = \mathrm{Ker}\,(G \text{ on } V)$. If W is an irreducible submodule of V^L, then W^σ is irreducible and $C = \mathrm{Ker}\,(G \text{ on } W^\sigma)$ for all $\sigma \in \mathcal{H}$.

Proof (a) [26], V, 13.2 and V, 13.3.

(b) Clear from definitions.

q.e.d.

I.2.11 Lemma Suppose that W is an irreducible LG-module with character χ such that $K(\chi) = K$. Then there exists a unique (up to isomorphism) irreducible KG-module V such that all submodules of V^L are isomorphic to W, and V affords the character $\varphi = m\chi$ for some integer m with $m \mid |L : K|$.

Proof Let V be an irreducible KG-submodule of W_K. By I.2.4 and I.2.6 the module V is uniquely determined (up to isomorphism) with the property that W is isomorphic to an irreducible LG-submodule of V^L. Let $\sigma \in \mathcal{H}$. By I.2.10 there is a submodule of V^L isomorphic to W^σ. Now, the character of W^σ is $\chi^\sigma = \chi^{\sigma|K(\chi)}$. But since by hypothesis $K = K(\chi)$ it follows that $\chi = \chi^\sigma$ and so by I.2.8 the LG-modules W and W^σ are isomorphic. The rest follows from [27], 9.18 (c).

q.e.d.

I.2.12 Definition (a) Let V be an irreducible KG-module. Then V is said to be absolutely irreducible if V^F is irreducible for all extension fields F of K. An irreducible K-character χ is absolutely irreducible if the KG-module affording χ is absolutely irreducible. The field K is a splitting field for G if every irreducible KG-module is absolutely irreducible.

(b) Let W be an irreducible LG-module with character χ and let V be an irreducible KG-module such that W appears as a direct summand of V^L. The multiplicity of W as a direct summand in a given direct decomposition of V^L into irreducible LG-modules is called the Schur L-index of W (or χ) over K, and is denoted $s_{L:K}(\chi)$. If L is a splitting field for G we talk of the Schur index of W (or χ) over K and write $s_K(\chi)$. In this case it is well-known that $s_K(\chi)$ is independent of choice of splitting field L.

(c) Let D be a representation of G. We set that D is realizable over K if there is a KG-module affording the representation D. Similarly, if χ is a class function on G then we say that χ is realizable as a K-character of G if there is a KG-module affording χ as its character.

The following theorem of Richard Brauer is of great interest and even greater inportance:

I.2.13 Theorem (R. Brauer [1]) Let D be an absolutely irreducible representation (i.e. a module affording D is absolutely irreducible) of G and let ε be a primitive $|G|$ - th root of unity. Then D is realizable over K (ε).

I.2.14 Theorem Let V be an irreducible KG-module, let W be an irreducible LG-module of V^L and let χ be the character afforded by W.

(a) If $L = K(\chi)$, then $s_{L:K}(\chi) = 1$.

(b) Let M be a Galois field extension of L, let U be an irreducible MG-module of W^M and let Ψ be the character afforded by U. Then

$$s_{M:K}(\Psi) = s_{M:L}(\Psi)\, s_{L:K}(\chi).$$

(c) $s_{L:K}(\chi) = s_{L:K(\chi)}(\chi)$.

(d) If $\sigma \in \mathcal{H}$, then $s_{L:K}(\chi) = s_{L:K}(\chi^{\sigma})$.

(e) If L is a field of non-zero characteristic, then

$$s_{L:K}(\chi) = 1.$$

called the Schur L-index of W (or χ) over K,* and is denoted $s_{L:K}(\chi)$. If L is a splitting field for G we talk of the Schur index of W (or χ) over K and write $s_K(\chi)$. In this case it is well-known that $s_K(\chi)$ is independent of choice of splitting field L.

(c) Let D be a representation of G. We set that D is realizable over K if there is a KG-module affording the representation D. Similarly, if χ is a class function on G then we say that χ is realizable as a K-character of G if there is a KG-module affording χ as its character.

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I.2.14 Theorem Let V be an irreducible KG-module, let W be an irreducible LG-module of V^L and let χ be the character afforded by W.

(a) If $L = K(\chi)$, then $s_{L:K}(\chi) = 1$.

(b) Let M be a Galois field extension of L, let U be an irreducible MG-module of W^M and let Υ be the character afforded by U. Then

$$s_{M:K}(\Upsilon) = s_{M:L}(\Upsilon)\, s_{L:K}(\chi).$$

(c) $s_{L:K}(\chi) = s_{L:K(\chi)}(\chi)$.

(d) If $\sigma \in \mathcal{H}$, then $s_{L:K}(\chi) = s_{L:K}(\chi^{\sigma})$.

(e) If L is a field of non-zero characteristic, then

$$s_{L:K}(\chi) = 1.$$

Proof (a) Appealing to V, 13.2 in [25] we have

$$\text{(1)} \quad V^L = \sum_{\sigma \in \mathcal{H}} W^\sigma.$$

If $\sigma \in \mathcal{H}$ and $W \cong W^\sigma$ then by I.2.8 we have $\chi = \chi^\sigma$. But then σ must be the trivial automorphism of $L = K(\chi)$ over K. Therefore the summands in (1) are pairwise non-isomorphic and the sum must therefore be direct. Part (a) now follows.

(b) The identity follows easily from I.2.6 and the fact that $V^M \cong (V^L)^M$.

(c) This is immediate from (b) and (a).

(d) By the proof of (a), we have

$$V^{K(\chi)} = \bigoplus_{\rho \in \mathrm{Gal}(K(\chi):K)} U^\rho$$

where U is an irreducible submodule of $V^{K(\chi)}$ having W as an irreducible submodule of U^L, which is itself a submodule of V^L. By I.2.11 and (a), every irreducible submodule of V^L isomorphic to W is contained in U^L - in fact, U^L is the homogeneous component of V^L corresponding to W. Since the homogeneous component corresponding to W^σ is $(U^L)^\sigma \cong (U^{\sigma'})$, where $\sigma' = \sigma \big|_{K(\chi)}$, the result clearly follows.

(e) Let M be a splitting field for G with $L \subseteq M$. By I.2.13 such M can be found as a Galois extension of L. By [27], 9.21 (b) and using (c) above we have

$$1 = s_K(\psi) = s_{M:L}(\psi).\, s_{L:K}(\chi)$$

where ψ is the character of an irreducible component of W^M. Obviously (e) follows from this identity.

q.e.d.

In the proof of parts (a) and (d) of I.2.14 we have in essence proved that the Galois conjugates of W over K (in the notation of that theorem) are completely determined by the Galois conjugates of U over K. In other words, if W is actually a submodule of U^L and ρ is an element of $\mathcal{H}$ then W^ρ is, up to isomorphism, the unique irreducible summand of $(U^{\rho'})$, where $\rho' = \rho|_{K(\chi)}$ and every Galois conjugate of W over K appears in this way. Thus if $\tau \in \mathrm{Gal}\,(K(\chi) : K)$ and U is an irreducible submodule of W_K, then W^τ will be taken to be that Galois conjugate (again, up to isomorphism) of W over K corresponding to an irreducible direct summand of $(U^\tau)^L$. These remarks are formalised by the main result of this section:

<u>I.2.15 Theorem</u> Let V be an irreducible KG-module, and let W be an irreducible submodule of V^L affording the character χ. If $\mathcal{G} = \mathrm{Gal}\,(K(\chi) : K)$ then

$$V^L \cong s_{L:K}(\chi) \cdot \bigoplus_{\rho\in\mathcal{G}} W^\rho .$$

Further, for distinct $\sigma, \rho \in \mathcal{G}$ the LG-modules W^σ and W^ρ are not isomorphic.

<u>Proof</u> The isomorphism follows directly from Lemma I.2.10 and Theorem I.2.14 (d). By I.2.14 the $K(\chi)$ G-module $V^{K(\chi)}$ is a direct sum of pairwise non-isomorphic irreducible $K(\chi)$ G-modules. Thus, if U is a direct summand of $V^{K(\chi)}$ such that W is a direct summand of U^L it follows firstly that $U^\sigma \not\cong U^\rho$, and then from I.2.6 and I.2.11 that $W^\sigma \not\cong W^\rho$.

q.e.d.

The promised full description of W_K can now be easily proved.

<u>I.2.16 Theorem</u> Let χ be the character afforded by an irreducible LG-module W and let V be an irreducible KG-module such that W is isomorphic to a direct summand V^L. Then

(a) $\dim_K V = s_{L:K}(\chi)\ |K(\chi) : K|\ \dim_L W$.

(b) $W_K = \bigoplus_{i=1}^{m} V$,

where $m = |L : K(\chi)| / s_{L:K}(\chi)$.

(c) If $s_{L:K}(\chi) = 1$ and $K = K(\chi)$, then $W \cong V^L$.

(d) If $s_{L:K}(\chi) = |L : K(\chi)|$, and in particular if $L = K(\chi)$, then $W_K \cong V$.

Proof (a) Let $\mathcal{G} = \mathrm{Gal}(K(\chi) : K)$. Appealing to I.2.15 we obtain

$$\dim_K V = \dim_L V^L = s_{L:K}(\chi).|\mathcal{G}|.\dim_L W$$

$$= s_{L:K}(\chi).|K(\chi) : K|.\dim_L W.$$

(b) By I.2.4, the KG-module W_K is a direct sum of copies of V. Since by part (a) of the same theorem we have $\dim_K W_K = |L : K| \dim_L W$, it follows from (a) that

$$\dim_K W_K = \frac{|L : K(\chi)|}{s_{L:K}(\chi)} . \dim_K V.$$

Part (b) is now clear.

(c) Suppose that $K = K(\chi)$ and $s_{L:K}(\chi) = 1$. Then we conclude from (a) that

$$\dim_K V = \dim_L V^L = \dim_L W.$$

(d) If $s_{L:K}(\chi) = |L : K(\chi)|$, then it follows from (b) that $W_K \cong V$.

q.e.d.

These results are to be applied in the sequel under the hypothesis that char $L \neq 0$ - a premise which in view of I.2.14 (e) obviously simplifies matters greatly. The following lemma demonstrates this observation, a full generalisation ([26], V, 14.11(b)) being less convenient than one would hope for.

I.2.17 Lemma Let χ be an irreducible L-character and suppose that char $L \neq 0$. Then $K(\chi)$ is the unique field extension of K of smallest degree over which χ is realisable.

Proof Obviously, any field over which χ is realizable and contains K must contain K (χ). Let W be an irreducible LG-module affording the character χ and let U be an irreducible K (χ) G-submodule of $W_{K(\chi)}$. By I.2.11 the character afforded by U is $m\chi$ for some m, whilst by I.2.14 (e) and I.2.16 (c) we have $W \cong U^L$. Evidently, then, $m = 1$ and the result is proven.

q.e.d.

I.2.18 Lemma Let K = GF (p^f), let V be an irreducible KG-module, let W be an irreducible submodule of V^L and let χ be the character afforded by W. If ψ is an absolutely irreducible character of V (see I.3.2) and $L \subseteq K(\psi)$, then $L = K(\chi)$. In particular, $W_K \cong V$ by I.2.16 (a).

Proof Without loss of generality, there is an irreducible K (ψ) G-submodule Y of $W^{K(\psi)}$ such that Y affords ψ. Certainly K $(\chi) \subseteq$ L. By I.2.17 and I.2.16 (c) there is an irreducible K (χ) G-module W_o such that $W_o{}^L \cong W$. By I.2.15 we have

(1) $\dim_{K(\chi)} W_o = \dim_{K(\psi)} W_o{}^{K(\psi)} = |K(\psi) : K(\chi)| \dim_{K(\psi)} Y$

and similarly,

(2) $\dim_L W = \dim_{K(\psi)} W^{K(\psi)} = |K(\psi) : L| \dim_{K(\psi)} Y,$

whilst

(3) $\dim_{K(\chi)} W_o = \dim_L W_o{}^L = \dim_L W.$

Thus comparing (1) and (2), bearing (3) in mind, yields

$$|K(\psi) : K(\chi)| = |K(\psi) : L| .$$

Since $K(\chi) \subseteq L \subseteq K(\psi)$, we deduce that $L = K(\chi)$.

q.e.d.

Armed with the weaponry developed in this section, we are equipped to launch a serious attack on the problems of applying Theorem I.1.4.

3. Applying Clifford theory in non-zero characteristic

Again in this section G will denote a group and L will be a Galois field extension of the field K. Our first result is easily proved, as inspection of the relevant definitions will bear out.

I.3.1 Lemma Let N be a subgroup of G.

(a) If V is an irreducible KG-module, then

$$(V|_N)^L = (V^L)|_N.$$

(b) If W is an irreducible LG-module, then

$$(W|_N)_K = (W_K)|_N.$$

(c) If U is an irreducible LN-module, then $(U^G)_K = (U_K)^G$.

Consider the following situation: let V be an irreducible KG-module and let N be a normal subgroup of G. Suppose that $V|_N$ is homogeneous. In order to apply I.1.4 we need to know whether an irreducible submodule, U say, of $V|_N$ is absolutely irreducible. If this is not the case, then we are, of course, tempted to employ the techniques and results of Section 2. However, it is now not clear that the irreducible submodules of V^L are again homogeneous on restriction to N. The corresponding problem in the non-homogeneous case is easily solved, but first we wish to extend the notation that will be fixed in this section in order to simplify the statement of such results. Therefore, from now on N will be a normal subgroup of G and U will be an irreducible N-submodule of an irreducible KG-module V. Further, if F is a Galois field extension of K which is a splitting field for G and all its subgroups (and such F exists by I.2.13) then χ will be the character afforded by an irreducible FG-submodule of V^F and φ

the character afforded by an irreducible FN-submodule of U^F.

I.3.2 Definition An absolutely irreducible component of V is an irreducible FG-submodule of V^F, where F is a field extension of K which is a splitting field for G. An absolutely irreducible character of V is the character of an absolutely irreducible component of V.

I.3.3 Theorem Suppose that $V\big|_N$ is not homogeneous and let W be an irreducible submodule of V^L. Then $W\big|_N$ is not homogeneous. If $|G : N| = q$, a prime, then $K(\varphi) = K(\chi)$.

Proof Suppose that $W\big|_N$ is homogeneous. Then by I.2.4 (b) and I.2.6, the KN-module $(W\big|_N)_K$ is a direct sum of isomorphic irreducible KN-modules. On the other hand, by I.2.4 (b) and I.2.6 again, W_K is a direct sum of copies of V and so by the hypothesis on V, the restriction of W_K to N cannot be homogeneous. Since $W_K\big|_N = (W\big|_N)_K$, this is a contradiction. Therefore our initial assumption must have been false, proving the first part of the theorem.

We now assume that $|G : N| = q$, a prime, and let M be a Galois extension of K which is a splitting field for both G and N. Choose an irreducible MG-submodule X of V^M which affords the character χ and let Y be an irreducible submodule of $X\big|_N$ which affords the character ψ say. By the above, $X\big|_N$ is not homogeneous, and since N is a maximal subgroup of G it follows that Y is a homogeneous component of $X\big|_N$ and that $X \cong Y^G$. By [26], V, 16.3 then, we have $\chi = \psi^G$ where ψ^G is defined as follows: set

$$\psi_0(g) = \begin{cases} \psi(g) & \text{if } g \in N \\ 0 & \text{if } g \notin N \end{cases} \quad \text{for } g \in G, \text{and let } g_1, \ldots, g_q$$

be a transversal of N in G. Then $\psi^G(g) = \sum_{i=1}^{q} \psi_o(g^{g_i})$ for $g \in G$. Hence $\chi(g)$ is a sum of values of ψ for all $g \in G$ and so

(1) $K(\chi) \subseteq K(\psi)$.

Set $F = K(\psi)$ and let T be an irreducible submodule of V^F such that X is an irreducible submodule of T^M. Such T exists as $V^M = (V^L)^M$ Since $T^M\big|_N = (T\big|_N)^M$, there is an irreducible submodule Z of $T\big|_N$ such that Y is isomorphic to an irreducible submodule of Z^M. From I.2.11 and I.2.16 (a) it follows that Z affords the character $s_K(\psi)\psi$ (for by I.2.14 (c) we have $s_F(\psi) = s_{K(\psi)}(\psi) = s_K(\psi)$). As in the proof of the first part of this theorem we see that $T \cong Z^G$, and so if η is the character afforded by T we have

$$\eta = (s_K(\psi)\psi)^G = s_K(\psi)\psi^G = s_K(\psi)\chi.$$

In particular, appealing to I.2.11 and I.2.16 (a) again, we obtain

$$\eta = s_K(\psi)\chi = s_F(\chi)\chi,$$

and hence

(2) $s_F(\chi) = s_K(\psi)$.

We now show that $s_{F:K(\chi)}(s_F(\chi)\chi) = 1$. Let $E = K(\chi)$. By I.2.16 (d) the EN-module Z_E is irreducible. Let S be an irreducible submodule of V^E such that T is isomorphic to an irreducible submodule of S^F. Clearly, as $S^F\big|_N = (S\big|_N)^F$ then Z_E is isomorphic to an irreducible submodule of $S\big|_N$. Using the first part of the theorem again we then have that $S \cong (Z_E)^G$, implying that $S \cong (Z^G)_E \cong T_E$, and in particular that T_E is an irreducible EG-module.

(4) $\dim_F T = q \cdot \dim_F Z$.

Using (3) and (4) together with I.2.16 (a) and I.2.4 (a) we obtain

$$\begin{aligned} s_{F:E}(s_F(\chi)\chi).\,|F:E|.\dim_F T &= \dim_E T_E \\ &= q.\ \dim_E Z_E \\ &= q.\ |F:E|.\dim_F Z \\ &= |F:E|.\dim_F T. \end{aligned}$$

Thus $s_{F:E}(s_F(\chi)\chi) = 1$, as claimed. Hence by I.2.14 (b) and (c) and (2) above, we infer that

$$\begin{aligned} (5)\quad s_K(\chi) &= s_E(\chi) \\ &= s_{F:E}(s_F(\chi)\chi).\,s_F(\chi) \\ &= 1.\ s_K(\psi). \end{aligned}$$

Finally, let R be the irreducible submodule of $V\big|_N$ such that Y is an irreducible submodule of R^M. Then, as before, $V \cong R^G$ and from I.2.16 (a) we have

$$\begin{aligned} (6)\quad s_K(\chi).\,|K(\chi):K|.\dim_M X &= \dim_K V \\ &= q\ .\ \dim_K R \\ &= q\ .\ s_K(\psi).\,|K(\psi):K|.\dim_M Y. \end{aligned}$$

But $X \cong Y^G$ and so $\dim_M X = q\ .\ \dim_M Y$. This fact together with (5) leads us to conclude from (6) that

$$|K(\chi):K| = |K(\psi):K|\ ,$$

whence, recalling (1), we have

(7) $K(\chi) = K(\psi)$.

Now, since φ is the character afforded by an irreducible submodule of $V^M\big|_N$, it follows from I.2.15 that there is an irreducible submodule Q of $X^\sigma\big|_N$ affording the character φ for some $\sigma \in \mathrm{Gal}(M:K)$. Clearly, $Q^{\sigma^{-1}}$ is isomorphic to an irreducible

($\underline{4}$) $\dim_F T = q \,.\, \dim_F Z$.

Using (3) and (4) together with I.2.16 (a) and I.2.4 (a) we obtain

$$\begin{aligned} s_{F:E}(s_F(\chi)\chi).\,|F:E|.\dim_F T &= \dim_E T_E \\ &= q.\,\dim_E Z_E \\ &= q.\,|F:E|.\dim_F Z \\ &= |F:E|.\dim_F T. \end{aligned}$$

Thus $s_{F:E}(s_F(\chi)\chi) = 1$, as claimed. Hence by I.2.14 (b) and (c) and (2) above, we infer that

($\underline{5}$) $s_K(\chi) = s_E(\chi)$

$$\begin{aligned} &= s_{F:E}(s_F(\chi)\chi)\,.\,s_F(\chi) \\ &= 1\,.\,s_K(\psi). \end{aligned}$$

Finally, let R be the irreducible submodule of $V\,\big|_N$ such that Y is an irreducible submodule of R^M. Then, as before, $V \cong R^G$ and from I.2.16 (a) we have

($\underline{6}$) $$\begin{aligned} s_K(\chi).\,|K(\chi):K|\,.\dim_M X &= \dim_K V \\ &= q\,.\,\dim_K R \\ &= q\,.\,s_K(\psi).\,|K(\psi):K|\,.\dim_M Y. \end{aligned}$$

But $X \cong Y^G$ and so $\dim_M X = q \,.\, \dim_M Y$. This fact together with (5) leads us to conclude from (6) that

$$|K(\chi):K| = |K(\psi):K| \;,$$

whence, recalling (1), we have

($\underline{7}$) $K(\chi) = K(\psi)$.

Now, since φ is the character afforded by an irreducible submodule of $V^M\,\big|_{N'}$ it follows from I.2.15 that there is an irreducible submodule Q of $X^{\sigma}\,\big|_N$ affording the character φ for some $\sigma \in \mathrm{Gal}\,(M:K)$. Clearly, $Q^{\sigma^{-1}}$ is isomorphic to an irreducible

MN-submodule of $X \big|_N$, and is therefore G-conjugate to the irreducible MN-submodule Y of $X \big|_N$. Let $g \in G$ with $\varphi^g = \psi^{\sigma^{-1}}$, the character afforded by $Q^{\sigma^{-1}}$. We clearly have

$$K(\psi) = K(\psi^{\sigma^{-1}}) = K(\varphi^g) = K(\varphi)$$

and in view of (7), the theorem is proven.

q.e.d.

Unfortunately, the analogous result does not hold when $V \big|_N$ is homogeneous, as an example will show.

<u>I.3.4 Example</u> Let G be the symmetric group of degree 3 and V a faithful and irreducible G-module over GF (2). It is well-known that dim V = 2. Let N = Alt(3), the normal subgroup of order 3 and index 2 in G. Since N cannot act trivially on any submodule of $V \big|_N$ it follows from I.1.7 that an irreducible submodule of $V \big|_N$ has dimension 2. Therefore $V \big|_N$ is irreducible, and in particular, is homogeneous. Now, set L = GF (4), and consider the LG- module V^L. Clearly, G is faithfully represented on the irreducible submodules of V^L (by I.2.10 (b)) and so, since G is not abelian and $\dim_L V^L = 2$, it must follow that V^L is an irreducible LG-module. Now, again appealing to I.1.7, we see that $V^L \big|_N$ decomposes into a sum of two 1-dimensional LN-modules, W_1 and W_2 say, which are, of course, absolutely irreducible. Since an involution in G inverts N, it is clear that W_1 and W_2 are not isomorphic LN-modules. Hence $V^L \cong (W_1)^G$ and $V^L \big|_N$ is not homogeneous.

With our general notation, therefore, we see that although $V \big|_N$ may be homogeneous, the irreducible components of V^L need not be homogeneous on restriction to N.

I.3.5 Definition Let $H \leqslant G$ and let Y be a KH-module. If $g \in N_G(H)$ then Yg is a KH-module in a natural way (c.f. I.1.1 (a)). The set

$$T = \{g \in N_G(H) : Yg \cong Y\}$$

is a subgroup of G containing H and is known as the stabilizer in G of Y, written $\mathrm{Stab}_G(Y)$. If ψ is the character afforded by Y, then for $g \in N_G(H)$ it is easy to see that the class function ψ^g defined by

$$\psi^g(h) = \psi(h^g)$$

for $h \in H$, is the character of Yg. Hence $\mathrm{Stab}_G(Y) = \{g \in N_G(H) : \psi^g = \psi\}$, the stabiliser in G of ψ, denoted $\mathrm{Stab}_G(\psi)$.

I.3.6 Theorem Recall the hypothesis introduced before definition I.3.2. We have a normal subgroup N of a group G, and an irreducible G-module V over some field K. Further, U is an irreducible N-submodule of $V|_N$, and χ and φ are absolutely irreducible characters of the G-module V and the N-module U, respectively. Suppose that char $K \neq 0$, and that $V|_N$ is homogeneous. Let T be the stabiliser in G of an absolutely irreducible component of U.

(a) T is the stabiliser in G of every absolutely irreducible component of U.

(b) $T \trianglelefteq G$, and the irreducible components of $(V|_T)^L$ are homogeneous on restriction to N.

(c) $V|_T$ is a direct sum of pairwise non-isomorphic irreducible KT-modules.

(d) If ψ is an absolutely irreducible character of $V|_T$, then $K(\chi,\varphi) = K(\psi)$.

(e) Let $\mathcal{N} = \mathrm{Gal}(K(\varphi) : K)$. There is a homomorphism $\alpha : G \to \mathcal{N}$ with Kernel T such that $\varphi^g = \varphi^{(g)\alpha}$ for all $g \in G$. In particular,

$G' \leqslant T$.

Proof Parts (a), (b), and (c) are proved first. We assume that L is a splitting field for G and all its subgroups. Let W be an irreducible component of V^L which affords the character χ and let Y be a submodule of $W|_N$. Without loss of generality, we may assume that Y affords the character φ. We can also assume that

(2) $W|_N$ is not homogeneous.

If (2) fails, then (a), (b) and (c) are true with $N = G$.

We fix some notation: let

$$W|_N = X_1 \oplus \ldots \oplus X_r$$

be the decomposition of $W|_N$ into homogeneous components. For $1 \leqslant i \leqslant r$ let Y_i be an irreducible LN-submodule of X_i and let n be the multiplicity of Y_i in X_i. For convenience, we assume that $Y \cong Y_1$ and that $T = \mathrm{Stab}_G(Y)$. Clifford's theorem tells us that X_1 is an irreducible LT-module and that $W \cong ((X_1)_T)^G$ - hence $r = |G : T|$. We set $\mathcal{G} = \mathrm{Gal}(K(\chi) : K)$ and $\mathcal{N} = \mathrm{Gal}(K(\varphi) : K)$.

Choose a transversal $\{g_1, \ldots, g_r\}$ of T in G such that $g_1 = 1$ and for $1 \leqslant i \leqslant r$,

(3) $X_i \cong X_1 g_i$.

In particular,

(4) $Y_i \cong Y g_i$ as LN-modules.

Furthermore, since $V|_N$ is homogeneous and $(V|_N)^L = V^L|_N$, the irreducible LN-submodules of $V^L|_N$ form a Galois conjugacy class. Hence for $1 \leqslant i \leqslant r$ there is an element $\sigma_i \in \mathcal{N}$ such that

(5) $Y_i \cong Y^{\sigma_i}$.

From (5) it follows that the stabiliser in G of Y_i must again be T. For if $h \in T$ we have

$$Y_i \cong Y^{\sigma_i} \cong (Yh)^{\sigma_i} \cong Y^{\sigma_i}h \cong Y_i\, h,$$

and so certainly $T \leqslant \mathrm{Stab}_G\,(Y_i)$. Since an analogous argument shows that $\mathrm{Stab}_G\,(Y_i) \leqslant T$, we have $T = \mathrm{Stab}_G\,(Y_i)$, as claimed. Notice also that it follows from (4) that $T = \mathrm{Stab}_G\,(Y_i) = T^{g_i}$. For if $h \in T$ we have

$$Y_i \cong Yg_i \cong Yhg_i \cong Yg_i h^{g_i} \cong Y_i h^{g_i}$$

implying that $T^{g_i} \leqslant \mathrm{Stab}_G\,(Y_i)$. Again, an analogous argument shows that $\mathrm{Stab}_G\,(Y_i)^{g_i^{-1}} \leqslant T$, proving our claim. Hence

$$T \trianglelefteq G.$$

We wish to show that T is the stabilizer of every irreducible LN-submodule of $V^L\big|_N$. Let Y_o be an arbitrary LN-submodule of V^L. Then by I.2.15 there is an element ρ of $\mathcal{H} = \mathrm{Gal}\,(L : K)$ such that Y_o appears as an irreducible submodule of $W^\rho\big|_N$. Indeed, we observe that

$$W^\rho = X_1^\rho \oplus \ldots \oplus X_r^\rho$$

is a decomposition of W^ρ into homogeneous LN-modules and hence $Y_o \cong Y_j^\rho$ for some $j \in \{1, \ldots, r\}$. The argument used previously shows that $\mathrm{Stab}_G\,(Y_o) = \mathrm{Stab}_G\,(Y_j) = T$ as required. This proves (a).

Again for $\rho \in \mathcal{H}$, we have that

$$((X_1)_T)^G \cong W, \text{ and}$$

$$((X_1^\rho)_T)^G \cong W^\rho.$$

Appealing, therefore, to [7], 45.6 (the condition that char $K \nmid |G|$ is redundant here since $T \trianglelefteq G$) we see that $W = W^\rho$ if and only if

$W^{\varphi} \big|_T$ has an irreducible submodule isomorphic to X_1. Because by I.2.14 (e) and I.2.15 the irreducible submodules of V^L are pairwise non-isomorphic, we therefore deduce that the irreducible submodules of $(V^L) \big|_T$ are pairwise non-isomorphic. Because $V^L \big|_T \cong (V \big|_T)^L$, it further follows that the irreducible submodules of $V \big|_T$ are pairwise non-isomorphic. This proves (c), and (b) is proven when L is a splitting field for G and all its subgroups.

Let now L_o be an arbitrary Galois extension of K. By suitable choice of L we may assume that $K \subseteq L_o \subseteq L$. Let X_o be an irreducible $L_o T$-submodule of $(V \big|_T)^{L_o}$, and suppose that $X_o \big|_N$ is not homogeneous. Then I.3.3 forces the irreducible submodules of $(X_o)^L$ to be not homogeneous on restriction to N. However, $X_o{}^L$ is a submodule of $((V \big|_T)^{L_o})^L \cong (V \big|_T)^L$, and we have already shown that the irreducible submodules of $(V \big|_T)^L \cong (V^L) \big|_T$ are homogeneous on restriction to N. This contradiction proves (b) completely.

We prove (d) next. First notice that since $W \cong ((X_1)_T)^G$ then $\chi = \psi^G$ and so

(6) $K(\chi) \subseteq K(\psi)$.

Suppose that $K(\varphi) \not\subseteq K(\psi)$ and let X be an irreducible $K(\psi)$ T - module such that $X^L \cong X_1$ (obviously $K(\psi) \subseteq L$, and such X exists by I.2.17). By (b), the restriction of X to N is homogeneous. By I.2.16 (d), $X^{K(\psi,\varphi)}$ is irreducible, and again by (b), $X^{K(\psi,\varphi)} \big|_N$ is homogeneous. However, because of our assumption that $K(\varphi) \not\subseteq K(\psi)$, it follows from I.2.17 that the irreducible submodules of $(X \big|_N)^{K(\psi,\varphi)}$ are absolutely irreducible and cannot be pairwise isomorphic, since $K(\psi,\varphi)$ is the unique smallest extension field of $K(\psi)$

over which the corresponding absolutely irreducible representations are realizable. Since $(X|_N)^{K(\psi,\varphi)} \cong X^{K(\psi,\varphi)}|_N$ this is a contradiction, proving that $K(\varphi) \subseteq K(\psi)$.

Thus with (6) we have

$$K(\varphi,\chi) \subseteq K(\psi).$$

Set $E = K(\varphi,\chi)$ and suppose that $E \neq K(\psi)$. Let W_o be an irreducible EG-submodule of V^E such that $W_o^L \cong W$ (again we can assume that $E \subseteq L$ and use I.2.17) and let X_o be an irreducible submodule of $W_o|_T$ such that X_1 is isomorphic to an irreducible submodule of X_o^L. Since $K(\psi) \not\subseteq E$ we have by I.2.17 that X_o^L is reducible. By (b) we have $X_o|_N$ is homogeneous and by I.2.17 again, the irreducible submodules of $X_o|_N$ are absolutely irreducible. Hence $X_o^L|_N$ is homogeneous. However, X_o^L is an LT-submodule of the irreducible LG-module $(W_o)^L \cong W$, and since by definition X_1 is a homogeneous component of $W|_N \cong W_o^L|_{N}$, we must have that $X_1 \cong X_o^L$. But X_1 is an irreducible LT-module, contradicting the reducibility of X_o^L. Hence the supposition that $E \neq K(\psi)$ must have been false, proving (d).

It remains to prove (e). We define a map $\alpha : G \to \mathcal{N}$ as follows: recall the definitions of σ_i and g_i for $1 \leq i \leq r$ as in (5) and (3) above. For $g \in G$ there exists $t \in T$ such that $g = tg_i$ for some uniquely determined $i \in \{1,\dots,r\}$. We set $(g)\alpha = \sigma_i$. It is clear that α is well-defined.

<u>α is a homomorphism</u>: let $g, g' \in G$. Then there exist $i, i' \in \{1,\dots,r\}$ and $t, t' \in T$ such that $g = tg_i$ and $g' = t'g_{i'}$. Then

$$gg' = tg_i t'g_{i'} = tt'^{g_i^{-1}} g_ig_{i'} \in T g_ig_{i'}.$$

Let $k \in \{1,\dots,r\}$ be such that $Tg_ig_{i'} = Tg_k$. Then there is an element t_o of T such that $gg' = t_o g_k$ and so $(gg')\alpha = (t_og_k)\alpha = \sigma_k$.

From (3) and (5) and the fact that $tt'^{g_i^{-1}} \in T$ we obtain

$$Y_1^{\sigma_k} \cong Y_1 g_k \cong Y_1 t_o g_k = Y_1 t t'^{g_i^{-1}} g_i g_{i'} \cong Y_1 g_i g_{i'}$$

$$\cong (Y_1^{\sigma_i}) g_{i'} \cong (Y_1 g_{i'})^{\sigma_i} \cong Y_1^{\sigma_{i'} \sigma_i} .$$

Since (by I.2.9. (a)) the Galois group $\mathcal{N}$ is abelian, we therefore have

$$Y_1^{\sigma_k} \cong Y_1^{\sigma_i \sigma_{i'}},$$

that is, $(gg')\alpha = \sigma_k = \sigma_i \sigma_{i'} = (tg_i\alpha)(t'g_{i'}\alpha) = (g)\alpha(g')\alpha$.

<u>$\operatorname{Ker}\alpha = T$:</u> Clearly σ_1 is the identity automorphism of $K(\varphi)$ over K, and it is immediate that $\ker\alpha = T$.

q.e.d.

It is theorem I.3.6 that helps us over the obstacles discussed in Section 1. There now follows a series of corollaries, all assuming the hypothesis of I.3.6.

<u>I.3.7 Corollary</u> The subgroup T is uniquely determined in G by properties (b) and (e) and $N \leq T$.

<u>Proof</u> Let T_o be another subgroup of G satisfying (b) and (e) with $N \leq T_o$. Since T_o has property (b) we must certainly have $T_o \leq T$. Let $\alpha_1 : G \longrightarrow \mathcal{N}$ be the homomorphism with kernel T_o and the property that $\varphi^g = \varphi^{(g)\alpha_1}$ for all $g \in G$, whose existence is guaranteed by (e). Assume that L is a splitting field for G, T and N (if in addition (d) is satisfied we need only take $L = K(\varphi)$.). Let W be an irreducible LG-submodule of V^L affording the character χ. Without loss of generality, there is an irreducible submodule Y of W affording the character φ. The property of α_1 implies that there are $|G : T_o|$ G-conjugates of Y. However, the definition of T is such that there are $|G : T|$ G-conjugates of Y. We therefore

infer that $|G : T| = |G : T_0|$. and hence that $T = T_0$.
q.e.d.

<u>I.3.8 Definition</u> The subgroup T defined in I.1.6 is called the <u>absolute stabiliser of</u> $V|_N$ <u>in G</u>, written $\mathrm{Abstab}_G(V|_N)$.

<u>I.3.9 Corollary</u> If N is abelian and $\mathrm{Ker}(N \text{ on } V) = 1$, then $T = C_G(N)$.

<u>Proof</u> This is immediate from (a).
q.e.d.

<u>I.3.10 Corollary</u> (a) Let M be an irreducible submodule of $V|_T$. Then $V \cong M^G$ if and only if $K(\chi) = K(\psi)$.

(b) If $N \triangleleft G$ and $V|_N$ is reducible, then $G = T$.

<u>Proof</u> (a) Let $S = \mathrm{Stab}_G(M)$. By I.3.6 (c) it is clear that M is a homogeneous component of $V|_S$, and so by Clifford's Theorem, $V \cong (M_S)^G$. In particular, $V|_T$ is a direct sum of $|G : S|$ irreducible KT-modules. Hence $(V|_T)^{K(\psi)}$ is a direct sum of $|G : S| . |K(\psi) : K|$ irreducible $K(\psi)$ T-modules. By I.3.6 (d) and I.2.15 we have that $V^{K(\psi)}$ is a direct sum of $|K(\chi) : K|$ absolutely irreducible $K(\psi)$ G-modules. Let W be such a $K(\psi)$ G-submodule of $V^{K(\psi)}$. Then because $T = \mathrm{Abstab}_G(V|_N)$ it follows that $W|_T$ is a direct sum of $|G : T|$ irreducible $K(\psi)$ T-modules, and hence that $V^{K(\psi)}|_T$ is a direct sum of $|G : T| . |K(\chi) : K|$ irreducible $K(\psi)$ T-modules. As $V^{K(\psi)}|_T \sim (V|_T)^{K(\psi)}$ we therefore have

(1) $|G : T| . |K(\chi) : K| = |G : S| . |K(\psi) : K|$.

Since by I.3.6 (d) we have $K(\chi) \subseteq K(\psi)$, the conclusion of part (a) of the corollary is immediate from (1).

(b) By I.3.6 (c) we cannot have $N = T$. Since $T \triangleleft G$ it follows that $G = T$.
q.e.d.

infer that $|G : T| = |G : T_o|$. and hence that $T = T_o$.
q.e.d.

<u>I.3.8 Definition</u> The subgroup T defined in I.1.6 is called the <u>absolute stabiliser of</u> $V|_N$ <u>in G</u>, written $\mathrm{Abstab}_G(V|_N)$.

<u>I.3.9 Corollary</u> If N is abelian and $\mathrm{Ker}(N \text{ on } V) = 1$, then $T = C_G(N)$.

<u>Proof</u> This is immediate from (a).
q.e.d.

<u>I.3.10 Corollary</u> (<u>a</u>) Let M be an irreducible submodule of $V|_T$. Then $V \cong M^G$ if and only if $K(\chi) = K(\psi)$.

(<u>b</u>) If $N \trianglelefteq G$ and $V|_N$ is reducible, then $G = T$.

<u>Proof</u> (<u>a</u>) Let $S = \mathrm{Stab}_G(M)$. By I.3.6 (c) it is clear that M is a homogeneous component of $V|_T$, and so by Clifford's Theorem, $V \cong (M_S)^G$. In particular, $V|_T$ is a direct sum of $|G : S|$ irreducible KT-modules. Hence $(V|_T)^{K(\psi)}$ is a direct sum of $|G : S| . |K(\psi) : K|$ irreducible $K(\psi)$ T-modules. By I.3.6 (d) and I.2.15 we have that $V^{K(\psi)}$ is a direct sum of $|K(\chi) : T|$ absolutely irreducible $K(\psi)$ G-modules. Let W be such a $K(\psi)$ G-submodule of $V^{K(\psi)}$. Then because $T = \mathrm{Abstab}_G(V|_N)$ it follows that $W|_T$ is a direct sum of $|G : T|$ irreducible $K(\psi)$ T-modules, and hence that $V^{K(\psi)}|_T$ is a direct sum of $|G : T| . |K(\chi) : K|$ irreducible $K(\psi)$ T-modules. As $V^{K(\psi)}|_T \cong (V|_T)^{K(\psi)}$ we therefore have

(<u>1</u>) $|G : T| . |K(\chi) : K| = |G : S| . |K(\psi) : K|$.

Since by I.3.6 (d) we have $K(\chi) \subseteq K(\psi)$, the conclusion of part (a) of the corollary is immediate from (1).

(<u>b</u>) By I.3.6 (c) we cannot have N [illegible] T. Since $T \trianglelefteq G$ it follows that $G = T$.
q.e.d.

infer that $|G : T| = |G : T_o|$. and hence that $T = T_o$.
q.e.d.

<u>I.3.8 Definition</u> The subgroup T defined in I.1.6 is called the <u>absolute stabiliser of</u> $V|_N$ <u>in G,</u> written $\mathrm{Abstab}_G(V|_N)$.

<u>I.3.9 Corollary</u> If N is abelian and Ker (N on V) = 1, then $T = C_G(N)$.

<u>Proof</u> This is immediate from (a).
q.e.d.

<u>I.3.10 Corollary</u> (a) Let M be an irreducible submodule of $V|_T$. Then $V \cong M^G$ if and only if $K(\chi) = K(\psi)$.

(b) If $N \trianglelefteq G$ and $V|_N$ is reducible, then $G = T$.

<u>Proof</u> (a) Let $S = \mathrm{Stab}_G(M)$. By I.3.6 (c) it is clear that M is a homogeneous component of $V|_T$, and so by Clifford's Theorem, $V \cong (M_S)^G$. In particular, $V|_T$ is a direct sum of $|G : S|$ irreducible KT-modules. Hence $(V|_T)^{K(\psi)}$ is a direct sum of $|G : S| . |K(\psi) : K|$ irreducible $K(\psi)$ T-modules. By I.3.6 (d) and I.2.15 we have that $V^{K(\psi)}$ is a direct sum of $|K(\chi) : K|$ absolutely irreducible $K(\psi)$ G-modules. Let W be such a $K(\psi)$ G-submodule of $V^{K(\psi)}$. Then because $T = \mathrm{Abstab}_G(V|_N)$ it follows that $W|_T$ is a direct sum of $|G : T|$ irreducible $K(\psi)$ T-modules, and hence that $V^{K(\psi)}|_T$ is a direct sum of $|G : T| . |K(\chi) : K|$ irreducible $K(\psi)$ T-modules. As $V^{K(\psi)}|_T \cong (V|_T)^{K(\psi)}$ we therefore have

(1) $|G : T| . |K(\chi) : K| = |G : S| . |K(\psi) : K|$.

Since by I.3.6 (d) we have $K(\chi) \subseteq K(\psi)$, the conclusion of part (a) of the corollary is immediate from (1).

(b) By I.3.6 (c) we cannot have $N = T$. Since $T \trianglelefteq G$ it follows that $G = T$.
q.e.d.

One can foresee immediately the usefulness of I.3.10 in inductive arguments. We can extend its range further after stating the following celebrated lemma of P. Hall and G. Higman.

I.3.11 Lemma (Hall, Higman [21]) (a) Suppose that $V|_N$ is homogeneous, that G/N is a p-group and $K = GF(p^f)$. Then $V|_N$ is irreducible.

(b) Suppose that $V|_N$ is homogeneous, that G/N is cyclic of prime order and that V is absolutely irreducible. Then $V|_N$ is irreducible.

Proof (a) [21], Lemma 2.2.3.

(b) Let $\overline{K}$ be the algebraic closure of K. Since V is absolutely irreducible, the $\overline{K}G$-module $V^{\overline{K}}$ is irreducible. Suppose that $V|_N$ is reducible. By I.3.10 (b) we have $G = \mathrm{Abstab}_G(V|_N)$, and hence that $V^{\overline{K}}|_N$ is homogeneous. By I.1.4, $V^{\overline{K}}|_N$ is a sum of n irreducible $\overline{K}N$-modules, where n is the degree of an irreducible projective $\overline{K}$-representation of G/N. By I.1.8 we have $n = 1$ and $V^{\overline{K}}|_N$ is irreducible. But because $V^{\overline{K}}|_N = (V|_N)^{\overline{K}}$, this contradicts our assumption that $V|_N$ is reducible. Hence $V|_N$ is irreducible as desired.

q.e.d.

I.3.12 Lemma Suppose that $|G/N| = q$, a prime, that $V|_N$ is homogeneous and that $K = GF(p^f)$. Let $V|_N = V_1 \oplus \ldots \oplus V_r$ be a decomposition of $V|_N$ into irreducible KN-modules.

(a) $r = 1$, q or $\deg_q p^{ef}$, where $e = |K(\varphi) : K|$.

(b) $r = |K(\chi) : K(\varphi)|$.

(c) If $r = q$, then $q \mid p^{fe} - 1$ and so $\deg_q p^f \mid e$.

(d) If $r = \deg_q p^{fe}$, then $q \nmid p^{fe} - 1$ and

$$|K(\chi) : K| = [|K(\varphi) : K|, \deg_q p^f].$$

<u>Proof</u> Clearly we may assume that $r > 1$ and hence, by I.3.11(a), that $q \neq p$. Observe that $G = \mathrm{Abstab}_G (V \mid_N)$ by I.3.10 (b). Let W be an irreducible submodule of $V^{K(\varphi)}$, and note that by I.3.6 (d) we have $K(\varphi) \subseteq K(\chi)$. We claim that W_K is irreducible. Let X be an irreducible submodule of $W^{K(\chi)}$. Since $V^{K(\chi)} \cong (V^{K(\varphi)})^{K(\chi)}$ and in view of I.2.17, the module X is Galois conjugate over K to an irreducible $K(\chi)$ G-module affording χ as character. Without loss of generality, therefore, we may assume that X affords the character χ. By I.2.16 (d) then, both $X_{K(\varphi)}$ and $X_K = (X_{K(\varphi)})_K$ are irreducible. But by I.2.4 (b) and I.2.6 we must have $X_{K(\varphi)} \cong W$ and hence W_K is irreducible, as claimed. Further, as $G = \mathrm{Abstab}_G (V \mid_N)$, it follows from I.2.16 (d) that $W \mid_N$ is homogeneous and that the irreducible $K(\varphi)$ N-submodules of W are irreducible when considered as KN-modules. In particular, $W \mid_N$ is the direct sum of r irreducible isomorphic $K(\varphi)$ N-modules. By I.1.4, r is the degree of an irreducible projective $K(\varphi)$-representation of $G / N \cong Z_q$. Part (a) now follows from I.1.8. With X as before, we have from I.3.11 (b) that $X \mid_N$ is irreducible. Since $W \mid_N$ is a sum of r absolutely irreducible $K(\varphi)$ N-modules it follows that $W^{K(\chi)}$ decomposes into a direct sum of r irreducible $K(\chi)$ G-modules, and (b) now follows from I.2.15.

(<u>c</u>) and (<u>d</u>) are corollaries of the proof of I.1.8 (the last part of (d) also uses (b)).

q.e.d.

<u>I.3.13 Examples</u> We show that all eventualities deemed possible in I.3.12 (a) actually happen.

(<u>a</u>) Let $S \cong \mathrm{Sym}(3)$ and let $A = F(S)$, an abelian subgroup of order 3. It is well-known that S is faithfully and irreducibly represented on a module V of dimension 2 over $GF(2)$. By I.1.7, we deduce that $V \big|_A$ is irreducible.

(b) Let Z be a cyclic group of order 4, and let A be the subgroup of order 2 in Z. By I.1.7, the dimension of a faithful irreducible Z-module V over GF (3) is 2. Clearly, $V \big|_A$ is homogeneous, and is the direct sum of two 1-dimensional A-modules. This situation corresponds to the case $r = q$ in Lemma I.3.12(a).

(c) Let a and b be coprime positive integers, and let Z be a cyclic group of order ab. By I.1.7, there is a faithful irreducible Z-module V of dimension $\deg_{ab} p$, where p is a prime with $p \nmid ab$, over GF (p). Let Z^* be the subgroup of Z of order a. Clearly $V \big|_{Z^*}$ is homogeneous, and an irreducible submodule of $V \big|_{Z^*}$ has dimension $\deg_a p$. Hence $V \big|_{Z^*}$ is a direct sum of $\deg_{ab} p / \deg_a p$ irreducible Z^*-modules. From I.1.6 (d) it follows that

$$\deg_{ab} p / \deg_a p = [\deg_a p, \deg_b p] / \deg_a p = \deg_b p^d,$$

where $d = \deg_a p$. If b is a prime and $\deg_b p \nmid d$, then this corresponds to the case $r = \deg_q p^e$ in I.3.12.

4. The degree of an irreducible modular representation of a soluble group

After a brief observation we state and prove the main result of this section. After some corollaries we will examine a few special cases in greater detail.

I.4.1 Lemma Let A be an abelian group of exponent a. If $N \trianglelefteq A$ and $\exp (A / N) < a$ then N contains a non-trivial characteristic subgroup of A.

Proof We shall presuppose Section 2 of Chapter II. Let $b = \exp (A / N)$ and for a positive integer n let $\mathfrak{A}(n)$ be the formation of abelian groups of exponent dividing n. Then N contains the $\mathfrak{A}(b)$-residual $A^{\mathfrak{A}(b)}$ of A and by hypothesis $A^{\mathfrak{A}(b)} \neq 1$. Since $A^{\mathfrak{A}(b)}$ char A we are finished.

q.e.d.

<u>I.4.2 Theorem</u> Let G be a soluble group, let $K = GF(p^f)$ and let V be an irreducible KG-module. In addition, let A be a normal abelian subgroup of G. Then there exist positive integers m and s with the following four properties:

(i) $\exp (A / \mathrm{Ker}\ (A \text{ on } V)) \mid m$;

(ii) $p \nmid m$;

(iii) $s \mid |G : A|$; and

(iv) $m\, s \mid |G|$,

such that

$$\dim_K V = s \deg_m p^f.$$

<u>Proof</u> We will argue by induction on $|G|$. If Ker (G on V) $\neq 1$ then we may apply the induction to V and G / Ker (G on V), yielding the result immediately. We therefore assume henceforth that V is faithful for G.. If G is abelian, then it follows from I.1.7 that G is cyclic and that $\dim_K V = \deg_{|G|} p^f$.

Set $|G| = \prod_{i=1}^{t} p_i^{a_i}$, where for $1 \leq i \leq t$ the p_i are distinct primes and the a_i are positive integers. Let b_i be an integer for $1 \leq i \leq t$ such that $|G : A| = \prod_{i=1}^{t} p_i^{b_i}$. It is easy to apply I.1.6 (c) and to see that there exist integers e_i such that $0 \leq e_i \leq b_i$ for $1 \leq i \leq t$ and

$$\dim_K V = \deg_{|G|} p^f = \prod_{i=1}^{t} p_i^{e_i} \cdot \deg_{|A|} p^f.$$

Setting $s = \prod_{i=1}^{t} p_i^{e_i}$ and $m = |A|$ it is clear that the theorem is proven in this case. Thus we may further assume that G is not abelian, so not simple, and we can find a non-trivial maximal normal subgroup N of G containing A. Let $|G : N| = q$, where q is, of course, a prime. We have several cases to consider.

<u>Case 1: $V|_N$ is irreducible</u> In this case the induction hypothesis applies to yield integers m and s satisfying the statement with N in place of G. Since these properties all carry over to G we have the result.

We therefore assume from now on that $V|_N$ is reducible. Applying Clifford's Theorem, we obtain a decomposition

$$V|_N = V_1 \oplus \dots \oplus V_r$$

of V into irreducible KN-submodules. By assumption, $r > 1$.

<u>Case 2: $V|_N$ is not homogeneous</u> The Clifford theory tells us that $r = q$ and that the V_i are the homogeneous components of $V|_N$. We may apply the induction hypothesis to N to yield integers m and s_1 such that

(i) $\exp (A / \mathrm{Ker} (A \text{ on } V_1)) \mid M$

(ii) $p \nmid m$

(iii) $s_1 \mid |N : A|$

(iv) $m s_1 \mid |N|$.

We see from I.4.1 that $\exp (A / \mathrm{Ker} (A \text{ on } V_1)) = \exp (A / \mathrm{Ker} (A \text{ on } V))$. For otherwise Ker (A on V) will contain a characteristic subgroup of A which, by virtue of the normality of A in G, must itself be normal in G. However, V is faithful for G and so no normal subgroup of G may have fixed points in V. This proves our claim, and shows that

$$\exp (A / \mathrm{Ker} (A \text{ on } V)) \mid m.$$

Now, $\dim_K V = q.\dim_K V_1 = q.s_1.\deg_m p^f$. Set $s = q\, s_1$. Since

$$s = q\, s_1 \mid q.|N : A| = |G : A|,$$

and

$$m\ s = m\ q\ s_1 \ \Big|\ q.|N| \ = \ |G|$$

the result follows.

<u>Case 3: $V\big|_N$ is homogeneous</u> Since $V\big|_N$ is reducible we deduce from I.3.11 (a) that $q \neq p$. By Corollary I.3.10 (b), we have that $G = \mathrm{Abstab}_G\,(V\big|_N)$. If φ is an absolutely irreducible character of V_1 and χ is an absolutely irreducible character of V, then from I.3.6 (d) we have

$$K\,(\varphi) \subseteq K\,(\chi).$$

Let $e = |K\,(\varphi) : K|$, so that $K\,(\varphi) = GF\,(p^{fe})$. From I.3.12 (a) we infer that $r = 1$, q or $\deg_q p^{fe}$. We have already eliminated the possibility that $r = 1$, and we are left with two further distinct cases.

<u>Case 3a: $r = q$</u>. This follows in the same way as Case 2, although we observe that because $V\big|_N$ is homogeneous, then $\mathrm{Ker}\,(A \text{ on} V_1) = \mathrm{Ker}\,(A \text{ on } V)$ and so the induction shows at once that (i) holds.

<u>Case 3b: $r = \deg_q p^{fe} > 1$.</u> Since $\deg_q p^{fe} > 1$, it follows that $\deg_q p^f > 1$ also. By I.3.12 (b) we have

$$\begin{aligned} |K\,(\chi) : K| &= |K\,(\varphi) : K|.|K\,(\chi) : K\,(\varphi)| \\ &= e\ .\ \deg_q p^{fe}. \end{aligned}$$

Clearly $\deg_q p^f \ \big|\ e \deg_q p^{fe}$, and so there is a subfield L of $K\,(\chi)$ containing K such that $|L : K| = \deg_q p^f$ (since $K\,(\chi) : K$ is a Galois extension with abelian Galois group). Let $d = \deg_q p^f$ and let Y be an irreducible LG-submodule of V^L. We claim that $Y\big|_N$ is irreducible. We show first that $L\,(\varphi) = L\,(\chi) = K\,(\chi)$. Clearly, since $K \subseteq L \subseteq K\,(\chi)$ and $K\,(\varphi) \subseteq K\,(\chi)$, we have $K\,(\varphi) \subseteq L\,(\varphi) \subseteq K\,(\chi) = L\,(\chi)$. By I.3.12 (d) we have

$$|K(\chi) : K| = [|K(\varphi) : K|, d].$$

But $K \subseteq K(\varphi) \subseteq L(\varphi)$ and $|L : K| = d$, so

$|K(\chi) : K| = [|K : K(\varphi)|, d] \,\Big|\, |L(\varphi) : K|$. It follows from this that $L(\varphi) = K(\chi)$ proving our claim.

It now follows from I.2.15 that $Y^{L(\varphi)}$ is a direct sum of $|K(\chi) : L|$ irreducible $K(\chi)$ G-modules. If $Y\big|_N$ is a direct sum of t isomorphic irreducible N-modules then $(Y\big|_N)^{L(\varphi)}$ is a direct sum of $t.|L(\varphi) : L| = t.|K(\chi) : L|$ irreducible $K(\chi)$ N-modules. But again since $G = \mathrm{Abstab}_G(V\big|_N)$, an irreducible LG-submodule W of $Y^{K(\chi)}$ is homogeneous on restriction to N (for $Y^{K(\chi)}$ is isomorphic to a submodule of $(V^L)^{K(\chi)} \cong V^{K(\chi)}$) and so by I.3.11 (b), $W\big|_N$ is irreducible. We therefore deduce that $|K(\chi) : L| = t.|K(\chi) : L|$, and so $t = 1$. Thus $Y\big|_N$ is irreducible, as claimed. By the inductive hypothesis there exist integers s and m_1 satisfying the statement of the theorem with N in place of G, Y in place of V and m_1 in place of m, such that

$$\dim_L Y = s.\deg_{m_1} p^{fd}.$$

Notice that Ker (A on V) = Ker (A on Y) by I.2.10 (b), so

$\exp(A / \mathrm{Ker}(A \text{ on } V)) \,\big|\, m_1$.

Since $L \subseteq K(\chi)$ we have from I.2.18 that $Y_K \cong V$, and so by I.2.4 (a) we have

$$\dim_K V = \dim_K Y_K = d \,.\, \dim_L Y = s\, d.\deg_{m_1} p^{fd}.$$

We finally apply I.1.6 (e) to get

$$\dim_K V = s.\deg_{[m_1, q]} p^f.$$

Set $m = [m_1, q]$. We have already noted that $\exp(A / \mathrm{Ker}(A \text{ on } V)) \,\big|\, m_1$, so evidently

(i) $\exp(A / \mathrm{Ker}(A \text{ on } V)) \,\big|\, m$.

Since $q \neq p$ and $p \nmid m_1$, we have

(ii) $p \nmid m$.

Since $s \mid |N : A|$ we certainly have

(iii) $s \mid |G : A|$,

and finally, because $m_1 s \mid |N|$ we get

$m.s \mid q.|N| = |G|$

and the theorem is proven.

q.e.d.

The first corollary is a special case of a theorem which may be traced back to being an easy consequence of a theorem by Swan ([31], Theorem 6) and was later proved using more direct methods by Dade [8].

<u>I.4.3 Corollary</u> (Swan, Dade). Let G be a (finite) soluble group, let K be a field of characteristic p, let A be an abelian normal subgroup of G and let V be an absolutely irreducible KG-module. Then

$$\dim_K V \mid |G : A|.$$

<u>Proof</u> Let D be the representation afforded by V. By I.2.13 we can realize D over a field which is the extension GF(p) by the $|G|$-th roots of unity. Therefore we may assume without loss of generality that K is this field, so that $K = GF(p^f)$ for some f. By I.4.2 there exist m and s satisfying $p \nmid m$, $m.s \mid |G|$ and $s \mid |G : A|$, and such that

$$\dim_K V = s.\deg_m p^f.$$

However, since $m \mid |G|$ and $p \nmid m$, our choice of K is such that $\deg_m p^f = 1$. Therefore

$$\dim_K V = s \mid |G : A|.$$

q.e.d

The second corollary is immensely important in Chapter III.

(ii) $p \nmid m$.

Since $s \mid |N : A|$ we certainly have

(iii) $s \mid |G : A|$,

and finally, because $m_1 s \mid |N|$ we get

$m.s \mid q.|N| = |G|$

and the theorem is proven.

q.e.d.

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Proof Let D be the representation afforded by V. By I.2.13 we can realize D over a field which is the extension GF(p) by the $|G|$-th roots of unity. Therefore we may assume without loss of generality that K is this field, so that $K = GF(p^f)$ for some f. By I.4.2 there exist m and s satisfying $p \nmid m$, $m.s \mid |G|$ and $s \mid |G : A|$, and such that

$$\dim_K V = s.\deg_m p^f.$$

However, since $m \mid |G|$ and $p \nmid m$, our choice of K is such that $\deg_m p^f = 1$. Therefore

$$\dim_K V = s \mid |G : A|.$$

q.e.d

The second corollary is immensely important in Chapter III.

I.4.4 Corollary Let G be a soluble group, let $K = GF(p^f)$ and let V be a faithful, irreducible G-module over K. Then there exist positive integers m and s with the following properties:

(i) $\exp(\mathrm{Soc}(G)) \mid m$;

(ii) $p \nmid m$;

(iii) m is square-free;

(iv) $ms \mid |G|$,

such that

$\dim_K V = s.\deg_m p^f$.

Proof This follows easily from the theorem by using I.1.6 (c) and (d).
q.e.d.

The last corollary is well-known.

I.4.5 Corollary Let G be a q-group for some prime q, let $K = GF(p^f)$ with $p \neq q$, and let V be an irreducible KG-module. Then

$$\dim_K V = q^a.\deg_q p^f$$

for some integer a with $q^a \mid |G|$.

Proof This is immediate from I.4.4.
q.e.d.

We can improve I.4.5 by employing I.3.6.

I.4.6 Theorem Let q be a prime, let G be a q-group and let $K = GF(p^f)$ with $p \neq q$. Let A be a maximal abelian normal subgroup of exponent q^a and let V be a faithful irreducible G-module over K. Then

$$\left[|G : A|, \deg_{q^a} p^f\right] \mid \dim_K V \mid |G : A|.\deg_{q^a} p^f.$$

Proof Note that since G is a q-group we have

$$A = C_G(A).$$

Consider first the possibility that $V|_A$ is homogeneous. By

Corollary I.3.9 we have $\mathrm{Abstab}_G (V \mid_A) = C_G(A) = A$, and so it follows from I.3.6 (c) that $V \mid_A$ is irreducible. Since V is faithful for G we deduce that V is faithful for A, and hence, from I.1.7, that A is cyclic and

$$\dim_K V = \deg_{q^a} p^f.$$

Lemma I.1.7 also implies that the smallest splitting field for A which contains K, call it L, has degree $d = \deg_{q^a} p^f$ over K. Since by I.3.6 (e) there is a monomorphism $\alpha : G / A \rightarrow \mathrm{Gal}(L : K) \cong Z_d$, we see that $|G : A| \mid \deg_{q^a} p^f$. Evidently then, the theorem is true in this case, for we have

$$\dim_K V = \deg_{q^a} p^f = [|G : A|, \deg_{q^a} p^f].$$

So now assume that $V \mid_A$ is not homogeneous and let

$$V \mid_A = V_1 \oplus \ldots \oplus V_r$$

be the decomposition of $V \mid_A$ into homogeneous components. Set $S = \mathrm{Stab}_G(V_1)$. Since Ker (A on V_1) may contain no normal subgroup of G, it follows from I.4.1 that $\exp(A / \mathrm{Ker}(A \text{ on } V_1)) = q^a$. Since $V_1 \mid_A$ is homogeneous, we may apply the above to yield

$$\dim_K V_1 = \deg_{q^a} p^f.$$

Notice that it again follows by I.3.9 and I.3.6 (e) that

$$|S : A| \mid \deg_{q^a} p^f.$$

Since $r = |G : S|$ we have that

$$|G : A| \mid \dim_K V = |G : S|.\deg_{q^a} p^f$$

and so clearly

$$\dim_K V \mid |G : A|.\deg_{q^a} p^f.$$

The result therefore follows.

q.e.d

I.4.7 Corollary Let G be a q-group of order q^{2n+1} having a maximal normal abelian subgroup A of index q^n. Then a faithful irreducible G-module over $K = GF(p^f)$ (if such exists) has dimension $q^n \deg_q p^f$.

Proof Let V be a faithful irreducible G-module over K (and so $p \neq q$) and let $q^a = \exp(A)$. Then by I.4.6 we have

$$(1) \quad [\,|G : A|\,, \deg_{q^a} p^f\,] \;\Big|\; \dim_K V \;\Big|\; |G : A| \deg_{q^a} p^f.$$

Clearly, if f is chosen to be sufficiently large, we have $\dim_K V = |G : A|$. We deduce, therefore, that an absolutely irreducible component of V has dimension $q^n = |G : A|$. Now, with the aid of I.1.6 (c), we infer from (1) that

$$\dim_K V = q^k . |G : A| . \deg_q p^f$$

for some k with $0 \leqslant k \leqslant a-1$. Recall Wedderburn's Theorem ([26], V, 4.5). Since $q \neq p$, the group ring KG is semisimple and so by that theorem we must have

$$(q^n)^2 q^k \deg_q p^f \leqslant |G|.$$

(We have used here the observation that an absolutely irreducible component of V has dimension q^n). Thus we can only have $k = 0$ or $k = 1$. If $k = 1$, then again using Wedderburn's Theorem we deduce that V is the unique (up to isomorphism) irreducible KG-module. Since, by hypothesis, V is faithful for G, this is a contradiction; for, of course, KG possesses an irreducible, trivial representation. Hence $k = 0$, and the corollary is proven.

q.e.d.

I.4.8 Corollary Let G be as in I.4.7 and let V be a faithful irreducible G-module over $K = GF(p^f)$. If χ is an absolutely irreducible character of V then $|K(\chi) : K| = \deg_q p^f$.

Proof Immediate from I.4.7 and I.2.16.

q.e.d

<u>I.4.9 Definition</u> Let r be a positive integer, let q be a prime with $q \nmid r$ and let V be a faithful irreducible Z_r-module over GF (q). Then E(r , q) denotes the semidirect product $[V] \cdot Z_r$. The group E (r , q) is soluble (but not nilpotent) and is non-abelian of order $r.q^d$, where $d = \deg_r q$.

<u>I.4.10 Lemma</u> Let p and q be distinct primes and let r be a positive integer with $q \nmid r$. Then a faithful irreducible E (r , q)-module over $K = GF(p^f)$ has dimension $[r, \deg_q p^f]$.

<u>Proof</u> Set G = E (r , q), let A = F (G) and let H be a complement of A in G. By Clifford's Theorem, $V|_A$ is completely reducible. Let Y be an irreducible submodule of $V|_A$, let $S = \mathrm{Stab}_G(Y)$ and let U be the homogeneous component of $V|_A$ which contains Y. Then $V \cong (U_S)^G$, and so $\dim_K V = |G : S|.\dim_K U$. Now, U is an irreducible KS-module and $U|_A$ is homogeneous. Since A is abelian and $A = F(G) \geqslant C_G(F(G))$, it follows from I.3.9 that $A = \mathrm{Abstab}_S (U|_A)$. By I.3.6 (c) then, we must have that $U|_A$ is irreducible, and so $U|_A = Y$. By I.1.7, $\dim_K U = \dim_K Y = \deg_q p^f$. Hence

$$\dim_K V = |G : S|.\deg_q p^f.$$

Let φ be an absolutely irreducible character of Y. By I.1.7 again we have $|K(\varphi) : K| = \deg_q p^f = |\mathcal{N}|$, where $\mathcal{N} = \mathrm{Gal}(K(\varphi) : K)$. By I.3.6 (e) then, $|S : A| \,\big|\, |\mathcal{N}|$ and so

$$|G : A| = |G : S|.|S : A| \,\Big|\, |G : S|.\deg_q p^f = \dim_K V.$$

Notice then that $[|G : A|, \deg_q p^f] \,\big|\, \dim_K V$. It remains, therefore, to check that $(r, \deg_q p^f) = |S : A|$. Let $t = (r, \deg_q p^f)$ and let T be the subgroup of H of order t. Since $t \mid \deg_q p^f$ and (by I.1.6 (a), (b)) $\deg_q p^f \mid q - 1$, we deduce that T acts linearly on A and $T \leqslant N_G(\mathrm{Ker}(A \text{ on } Y))$. Let

$A_2 = \mathrm{Ker}\,(A \text{ on } Y)$, and let A_1 be an irreducible T-module with $A = A_1 \oplus A_2$. Let $T = \langle x \rangle$. We wish to show that $Y \cong Yx$ as A-modules. Clearly it is sufficient to prove that $Y \cong Yx$ as A_1-modules. Now, T acts as a group of automorphisms on A_1, and we see that $A_1.T$ is isomorphic to a subgroup of the group Γ of semilinear transformations on Y (see [26], II, 3.10 for the definition of this group). Since Y is irreducible as a Γ-module, the modules Y and Y x must be isomorphic A_1-modules, as desired.

q.e.d.

I.4.11 Corollary Let p, q and r be as in the statement of I.4.10 and let χ be an absolutely irreducible character of a faithful irreducible $E\,(r\,,\,q)$-module over $GF\,(p^f)$. Then
$|K(\chi) : K| = \deg_q p^f / (r, \deg_q p^f)$.

Proof By I.4.10 and I.2.16 we have

$$|K(\chi) : K| = [r, \deg_q p^f] / r = \deg_q p^f / (r, \deg_q p^f).$$

q.e.d.

We conclude the chapter with some results about representations of direct products of groups.

I.4.12 Theorem (Fein [11], [12]) Let G_1 and G_2 be groups and let K be a field. For $i = 1, 2$ let V_i be an irreducible KG_i-module and let χ_i be an absolutely irreducible character of V_i. Then $V_1 \otimes_K V_2$ is a completely reducible $K(G_1 \times G_2)$-module in which the irreducible components appear with equal multiplicity and with the same dimension, namely

$$s_K(\chi_1\chi_2).\ |K(\chi_1\chi_2) : K|\ .\ \dim_{K(\chi_1\chi_2)} W_{K(\chi_1\chi_2)}$$

where W is an absolutely irreducible component of $V_1 \otimes_K V_2$.

A_2 = Ker (A on Y), and let A_1 be an irreducible T-module with $A = A_1 \oplus A_2$. Let $T = \langle x \rangle$. We wish to show that $Y \cong Y x$ as A-modules. Clearly it is sufficient to prove that $Y \cong Y x$ as A_1-modules. Now, T acts as a group of automorphisms on A_1, and we see that $A_1.T$ is isomorphic to a subgroup of the group Γ of semilinear transformations on Y (see [26], II, 3.10 for the definition of this group). Since Y is irreducible as a Γ-module, the modules Y and Y x must be isomorphic A_1-modules, as desired.

q.e.d.

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$|K(\chi) : K| = \deg_q p^f / (r, \deg_q p^f)$.

Proof By I.4.10 and I.2.16 we have

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$$s_K(\chi_1 \chi_2). \; |K(\chi_1\chi_2) : K| \; . \dim_{K(\chi_1\chi_2)} W_{K(\chi_1\chi_2)}$$

where W is an absolutely irreducible component of $V_1 \otimes_K V_2$.

Also, $|K(\chi_1\chi_2) : K| = [|K(\chi_1) : K|, |K(\chi_2) : K|]$ and if one of V_1 and V_2 is absolutely irreducible, then $V_1 \otimes_K V_2$ is irreducible. If both are absolutely irreducible, then $V_1 \otimes_K V_2$ is absolutely irreducible, and consequently, if char $K \neq 0$ and W_i is an (absolutely) irreducible submodule of $V_i^{K(\chi_i)}$, $i = 1, 2$, then $\dim_{K(\chi_1\chi_2)} W = \dim_{K(\chi_1)} W_1 \cdot \dim_{K(\chi_2)} W_2$.

<u>Proof</u> The first part of the statement is proved in [11], Theorem 2.2 Since $K(\chi_1)$ and $K(\chi_2)$ are both normal extensions of K contained in $K(\chi_1, \chi_2)$ there is a unique composite of them, and this composite is $K(\chi_1, \chi_2)$. Clearly

$$|K(\chi_1, \chi_2) : K| = [|K(\chi_1) : K|, |K(\chi_2) : K|].$$

If we show that $K(\chi_1, \chi_2) = K(\chi_1\chi_2)$, then the second part of the statement will follow. Certainly $K(\chi_1\chi_2) \subseteq K(\chi_1, \chi_2)$. Suppose that $K(\chi_1\chi_2) \subset K(\chi_1, \chi_2)$, and let W be an irreducible $K(\chi_1\chi_2)(G_1 \times G_2)$-submodule of $(V_1 \otimes_K V_2)^{K(\chi_1\chi_2)}$. By I.2.15 the $K(\chi_1, \chi_2)(G_1 \times G_2)$-module $W^{K(\chi_1, \chi_2)}$ must decompose into a homogeneous sum of irreducible $(G_1 \times G_2)$-modules. Because G_1 is centralised by G_2, the restriction $W^{K(\chi_1, \chi_2)}\Big|_{G_1}$ must similarly be homogeneous. On the other hand, $W\Big|_{G_1}$ is homogeneous, the components being isomorphic to a component of $V_1^{K(\chi_1\chi_2)}$. If $K(\chi_1) \not\subseteq K(\chi_1\chi_2)$ then, appealing to I.3.15 again, $(W\Big|_{G_1})^{K(\chi_1, \chi_2)}$ cannot be a homogeneous sum of irreducible $K(\chi_1, \chi_2)$ G -modules. This contradiction therefore forces $K(\chi_1) \subseteq K(\chi_1\chi_2)$. Similarly, $K(\chi_2) \subseteq K(\chi_1\chi_2)$, and we deduce that $K(\chi_1\chi_2) = K(\chi_1, \chi_2)$, as required.

The statements regarding the irreducibility and absolute irreduciblity follow from [11], Theorem 2.3 and [12], Proposition 2.1.

q.e.d.

The last theorem will be used in tandem with the following lemma.

<u>I.4.13 Lemma</u> (Förster [13], Lemma 1.8) With the notation of I.4.12 assume further that Ker $(G_i \text{ on } V_i) = 1$ for $i = 1, 2$. Let T be an irreducible submodule of $V_1 \otimes_K V_2$ and let $C = \text{Ker}\,(G_1 \times G_2 \text{ on } T)$. Then $C \leq Z(G_1) \times Z(G_2)$ and $C \cap (Z(G_i) \times 1) = 1$ for $i \in \{1, 2\}$. Further, $(C(G_1 \times 1)) \cap (1 \times G_2) \cong (C(1 \times G_2)) \cap (G_1 \times 1)$, and so T is faithful for $G_1 \times G_2$ if $(|Z(G_1)|, |Z(G_2)|) = 1$.

Chapter II

Chapter II. Finite soluble group theory

At the time of writing there are few texts which deal adequately with the fundamental theory of finite soluble groups. Perhaps the best general introduction available is by Gaschütz [19]; Cossey [6] and Huppert [26], Chapter VI are also of general interest although somewhat dated. The forthcoming book by K. Doerk and T. O. Hawkes [10] will provide by far the most comprehensive "state of the art" survey of finite soluble group theory. Their book will chart the course of the theory from "first principles" but can only be anticipated at present. Our aim here is only to introduce those concepts that are of use to us. For the rest of this thesis we will work under the following hypothesis:

All groups under consideration will be finite and soluble.

1. Chief factors and primitive groups

For the definition and elementary properties of soluble groups, the reader is referred to [26], I, Section 8.

We begin with a few of the more useful results about finite soluble groups.

II.1.1 Theorem Let G be a group.

(a) $C_G(F(G)) \leqslant F(G)$.

(b) $\Phi(G) < F(G)$.

(c) $F(G/\Phi(G)) = F(G)/\Phi(G)$.

Proof [26], III, 4.2 (b), (c) and (d).
q.e.d.

The hypothesis that G is soluble is not necessary to prove II.1.1(c). Nor is it necessary in II.1.2.

II.1.2 Theorem Let G be a group.

(a) (Gaschütz) $F(G)/\Phi(G)$ is the direct product of abelian minimal normal subgroups of $G/\Phi(G)$.

(b) If A is an abelian normal subgroup of G with $A \cap \Phi(G) = 1$, then there is a complement of A in G.

Proof (a) [26], III, 4.5.

(b) [26], III, 4.4.

q.e.d.

II.1.3 Corollary The minimal normal subgroups of $G/\Phi(G)$ are complemented in $G/\Phi(G)$.

II.1.4 Theorem Let N be a minimal normal subgroup of G with $N = C_G(N)$. Then there is a complement H of N in G with $\bigcap_{g \in G} H^g = 1$. The permutation representation of G on the cosets of H in G is a faithful representation of G as a primitive permutation group.

Proof [26], II, 3.3.

q.e.d.

II.1.5 Definition A group G having a self-centralizing minimal normal subgroup is called primitive. By convention, 1 is not a primitive group.

II.1.6 Theorem (Galois) Let G be a primitive group and let N be a minimal normal subgroup of G. Then N is the unique minimal normal subgroup in G. If H is a complement of N in G and p is the prime divisor of $|N|$, then $\dot{N}$ may be considered as a faithful irreducible G-module over GF (p) and $G \cong [N]\cdot H$. All complements of N in G are conjugate.

<u>Proof</u> [26] II, 3.2.
q.e.d.

<u>II.1.7 Lemma</u> Let G be a group, let $H <\cdot G$, and let N be a minimal normal subgroup of G. If $N \not\leqslant H$, then $N H = G$ and $N \cap H = 1$.

<u>Proof</u> Since $H <\cdot G$ it is obvious that $N H = G$. Suppose that $N \cap H \neq 1$. Since $N \trianglelefteq G$ we see that

$$H \leqslant N_G (N \cap H).$$

Since N is abelian, we therefore have

$$G = N H \leqslant N_G (N \cap H).$$

However, $N \cap H < N$ and N is a minimal normal subgroup of G. This contradiction proves the lemma.
q.e.d.

<u>II.1.8 Definition</u> Let G be a group and $H \leqslant G$.
Then

$$\mathrm{Core}_G (H) = \bigcap_{g \in G} H^g.$$

This subgroup of H is called the <u>normal core</u> of H in G; it is the largest normal subgroup of G contained in H.

<u>II.1.9 Theorem</u> Let G be a group and $N \trianglelefteq G$. Then G / N is a primitive group if and only if there is a maximal subgroup H of G such that $N = \mathrm{Core}_G (H)$.

<u>Proof</u> Let H be a maximal subgroup of G and set $N = \mathrm{Core}_G (H)$.

Clearly, H / N is a maximal subgroup of G / N with trivial normal core. Let M be a normal subgroup of G such that $N \leq M$ and M / N is a minimal normal subgroup of G / N. By II.1.7 we have $(M / N)(H / N) = HM / N = G / N$ and $M \cap H = N$. Since $C_{G/N}(M / N) \trianglelefteq G / N$, it follows that $C_{G/N}(M / N) \cap H / N \trianglelefteq H / N$ and so

$$G / N = (M / N)(H / N) \leq N_{G/N}(C_{G/N}(M / N) \cap H / N).$$

Since H / N has trivial core, we deduce that $C_{G/N}(M / N) \cap H / N = 1$, and hence

$$C_{G/N}(M / N) = M / N.$$

Thus G / N is primitive.

Suppose now that $N \trianglelefteq G$ and G / N is primitive. Let $H \lessdot G$ be such that $N \leq H$ and H / N is a complement of the unique minimal normal subgroup M / N of G / N. It is now an easy matter to check that $N = \mathrm{Core}_G(H)$.

q.e.d.

Thus primitive groups appear in a natural way. There is also a link between primitive quotients of a group and its chief factors.

II.1.10 Definitions Let G be a group.

(a) Let

$$G = G_0 > G_1 > \dots > G_s = 1$$

be a series of normal subgroups of G. This series is called a chief series of G if whenever $N \trianglelefteq G$ and $G_{i-1} \geq N \geq G_i$, then $N \in \{G_{i-1}, G_i\}$ for $i \in \{1, \dots, s\}$. A quotient G_{i-1} / G_i, $1 \leq i \leq s$, is called a chief factor of G.

(b) Let π be a set of primes and let M / N be a chief factor of G. Because of our hypothesis that G is soluble, M / N is an elementary abelian p-group for some prime p. If $p \in \pi$ we say that M / N is a π-chief factor of G. If $\pi = \{p\}$ we write that M / N is a p-chief factor.

(c) Let M / N be a chief factor of G and $H \leq G$. Then

$$C_H(M / N) = \{h \in H : N m^h = Nm \text{ for all } m \in M\}.$$

Evidently, $G / C_G(M / N)$ is isomorphic to a subgroup of Aut (M / N). We write $A_G(M / N) = G / C_G(M / N)$ and talk of the group of automorphisms induced on M / N in G.

(d) Let M / N be a chief factor of G. If $M / N \leq \Phi(G / N)$ then M / N is said to be Frattini in G. Otherwise M / N is said to be complemented.

II.1.11 Lemma Let G be a group and let

$$G = G_0 \trianglerighteq G \trianglerighteq \ldots \trianglerighteq G_n = 1$$

be a chief seies of G. Then

$$F(G) = \bigcap_{i=1}^{n} C_G(G_i / G_{i+1}).$$

Proof [26], III, 4.3.

q.e.d.

II.1.12 Lemma Let M / N be a complemented chief factor of a group G. Then M / N is a complemented subgroup of G / N and there is a maximal subgroup H of G such that

$$[M / N] \cdot A_G(M / N) \cong G / \mathrm{Core}_G(H).$$

In fact, $\mathrm{Core}_G(H) = C_H(M / N)$ ($= C_G(M / N) \cap H$).

Proof Since M / N is a minimal normal subgroup of G / N such that $M / N \nleq \Phi(G / N)$ it follows that $M / N \cap \Phi(G / N) = 1$. Applying II.1.2 (b) yields a complement H / N of M / N in G / N where H is a subgroup of G with $N \leq H$. We claim that

$\mathrm{Core}_G(H) = H \cap C_G(M/N)$. Obviously, $H \cap C_G(M/N) = C_H(M/N)$. Since $H \cap M = N$ we have $[M, \mathrm{Core}_G(H)] \leq N$, and so

$$\mathrm{Core}_G(H) \leq C_H(M/N).$$

Now, $C_H(M/N) \trianglelefteq H$. Also, $[M, C_H(M/N)] \leqslant N \leq C_H(M/N)$. Therefore $M \leqslant N_G(C_H(M/N))$ and we obtain

$$C_H(M/N) \trianglelefteq H\,M = G,$$

and so by the definition of normal core, we have $C_H(M/N) \leqslant \mathrm{Core}_G(H)$. This completes the proof of our claim that $C_H(M/N) = \mathrm{Core}_G(H)$.

Since $M \leqslant C_G(M/N)$ and $H\,M = G$ we have

$$H.C_G(M/N) = G.$$

Thus

$$H / \mathrm{Core}_G(H) = H / H \cap C_G(M/N) \cong G / C_G(M/N).$$

The result follows easily from this isomorphism.

q.e.d.

<u>II.1.13 Definition</u> If M/N is a complemented chief factor of a group G and $H <\cdot G$ is such that $N \leqslant H$ and H/N is a complement of M/N in G, then we call H a <u>complement</u> of M/N in G.

In view of the definitions, and II.1.2 (b), the next lemma is an obvious remark.

<u>II.1.14 Lemma</u> Let M/N be a chief factor of a group G. Then M/N is complemented in G if and only if there exists a complement to M/N in G/N.

A fundamental result concerning chief series is a theorem of Zassenhaus. Here we state a generalisation of that theorem due to Carter, Fischer and Hawkes.

<u>II.1.15 Theorem</u> (Carter, Fischer and Hawkes [3]) Let G be a group.

Given two chief series passing through $N \trianglelefteq G$, there is a one-to-one correspondence between the chief factors of the series below N, corresponding factors being G-isomorphic, such that the Frattini factors of one series correspond to the Frattini factors of the others.

Proof [3], Lemma 2.6.

q.e.d.

II.1.16 Lemma Let M / N be a chief factor of a group such that $\Phi(G) \leqslant N < M \leqslant F(G)$. Then M / N is complemented in G.

Proof By II.1.3, the minimal normal subgroups of $G/\Phi(G)$ are complemented, and by II.1.2 (a), $F(G)/\Phi(G)\ (= F(G/\Phi(G)))$ is a direct product of minimal normal subgroups of $G/\Phi(G)$. The result follows now from II.1.15.

II.1.17 Lemma Let G be a group and let p be a prime such that $p \mid |G|$ but $p \nmid |F(G)|$. Let S be a Sylow p-subgroup of G. Then there is a complemented chief factor of G between $\Phi(G)$ and $F(G)$ on which S acts non-trivially.

Proof By II.1.1 (c) we have $F(G)/\Phi(G) = F(G/\Phi(G))$, and so $p \nmid |F(G/\Phi(G))|$. We denote by $\overline{B}$ the image of a subgroup B of G under the natural homomorphism of G onto $G/\Phi(G)$. By II.1.1 (a), $C_{\overline{G}}(F(\overline{G})) \leqslant F(\overline{G})$ and so $\overline{S} \cap C_{\overline{G}}(F(\overline{G})) = 1$. By II.1.2 (a), the group $F(\overline{G})$ is a direct product of minimal normal subgroups of $\overline{G}$, and $\overline{S}$ must therefore act non-trivially on one of these minimal normal subgroups - $\overline{N}$ say, with $\Phi(G) < N \leqslant F(G)$. Then $N/\Phi(G)$ is a chief factor of G on which S acts non-trivially. By II.1.16, this chief factor is complemented.

q.e.d.

We finish Section 1 with two results concerning representations and primitive groups. The first is a construction due to Herr Dr Peter Förster, the essence of which is to be found in the proof of Lemma 5.6 in [14]. The proposition is an improvement of that construction which was pointed out by Dr Förster, to whom the author is most grateful.

II.1.18 Proposition Let G be a primitive group, let $V = \mathrm{Soc}\,(G)$ and let H be a complement of V in G. Let q be the prime divisor of $|V|$ and let K be a field of characteristic different from q. Then there exists an integer n such that there is a faithful irreducible module X over K for the group $[V \times \overset{n}{\cdots} \times V] \cdot H$ of dimension $d.|H|$, where d is an integer dividing the dimension of a non-trivial irreducible $K\,Z_q$-module.

Proof Clearly we may assume that $H \neq 1$. Let $|V| = q^l$. By a well-kno theorem in linear algebra, V considered as a vector space possesses exactly $a = (q^l - 1)/(q - 1)$ hyperplanes, $U_1, \ldots, U_a$ say. Let $V_1, \ldots, V_a$ be isomorphic copies of the GF (q) H-module V and for $i = 1, \ldots, a$ let $\varphi_i : V \longrightarrow V_i$ be a GF (q) H-isomorphism. Then $\varphi_1(U_1) \times \ldots \times \varphi_a(U_a)$ is a GF (q)-subspace of $\overline{V} = V_1 \times \ldots \times V_a$ of co-dimension a. For $1 \leq i \leq a$ let $U'_i = \varphi_i(U_i)$, let W_i be a complement of U'_i in V_i (considered as a GF (q)-space) and let w_i be a generator of W_i - that is, $\{w_i\}$ is a GF (q)-basis of W_i. Set $W = W_1 \times \ldots \times W_a$, a complement of $U'_1 \times \ldots \times U'_a$ in $\overline{V}$, and define a GF (q)-epimorphism $\Psi : W \longrightarrow K$ by

$$\Psi((w_1^{b_1}, \ldots, w_a^{b_a})) = b_1 + \ldots$$

Let $\overline{U}$ be the GF (q)-subspace $\langle U'_1 \times \ldots \times U'_a, \mathrm{Ker}\,\Psi \rangle$ of $\overline{V}$. Elementary linear algebra considerations show that $\mathrm{Ker}\,\Psi$ is of co-dimension 1 in W and so $\overline{U}$ is of co-dimension 1 in $\overline{V}$. Further

$\overline{U} \cap V_i = U'_i$ for $1 \leqslant i \leqslant a$.

The group H acts on $\overline{V}$ in an obvious way. Set $Q = [\overline{V}] \cdot H$ and let Z be the largest subgroup of $Z(H)$ of order dividing $q-1$. We claim that $N_Q(\overline{U}) = \overline{V}Z$. Let $T = N_Q(\overline{U})$ and $T_o = T \cap H = N_H(\overline{U})$. Obviously $\overline{V} \leqslant T$, and since elements of Z act linearly on $\overline{V}$ we even have $Z \leqslant T_o \leqslant T$. Thus

(1) $\overline{V}Z \leqslant T$.

Now, $T_o \leqslant N_H(\overline{U} \cap V_i) = N_H(U'_i)$ for $i = 1, \ldots, a$, and so, since φ_i is an H-isomorphism, T_o normalizes every hyperplane of V. Hence T_o acts linearly on V, yielding $T_o \leqslant Z(H)$ and $|T_o| \mid q-1$. Therefore $T_o \leqslant Z$ and $T = \overline{V}T_o \leqslant \overline{V}Z$. This, together with (1), proves that $T = \overline{V}Z$, as claimed. Hence $T \trianglelefteq Q$.

As Z acts faithfully on $\overline{V}/\overline{U}$ and since $\overline{V}/\overline{U}$ is evidently a complemented chief factor of T, it follows from II.1.12 that

$$[\overline{V}/\overline{U}] \cdot T / C_T(\overline{V}/\overline{U}) \cong T/\overline{U},$$

for $\overline{U}Z$ is a maximal subgroup of T complementing $\overline{V}/\overline{U}$ with normal core $\overline{U}$. This group is primitive and isomorphic to $E(|Z|, q)$. We can find an irreducible KT-module Y with dimension $d.|Z|$ and $\overline{U} = \mathrm{Ker}\,(T \text{ on } Y)$, where d is an integer dividing the dimension of a non-trivial irreducible $K Z_q$-module; if char $K \neq 0$ this follows from I.4.10, whilst if char $K = 0$ we appeal to [26], V, 18.4 and apply the techniques of Section 2 in Chapter I. We show that $X = Y^Q$ is an irreducible KQ-module. Suppose to the contrary. Then an application of Mackey's Theorem (see [26], V, 16.9 (a)) shows that there is a KQ-composition factor X_o of X such that $X_o\big|_T$ has a submodule isomorphic to Y. Let Y_o be the homogeneous component of $X_o\big|_T$ corresponding to Y. Since $T \trianglelefteq Q$ we can use Clifford's Theorem. Let $\overline{T} = \mathrm{Stab}_Q(Y)$.

Obviously $T \leqslant \dot{\overline{T}}$. On the other hand, it must follow that $\overline{T} \leqslant N_Q$ (Ker (T on Y)) = $N_Q(\overline{U}) = T$. Hence $T = \overline{T}$ and Y_o is isomorphic (as a KT-module) to Y. Since by Clifford's Theorem $X_o \cong (Y_o)^Q$, it follows that $X_o = X$, as claimed. Notice further that

$$\dim_K X = |Q : T|.\dim_K Y = |Q : T|.|Z|.d = d.|H| .$$

Finally we see that

$$\text{Ker } (Q \text{ on } X) = \bigcap_{h \in H} \overline{U}^h \leqslant \overline{V}.$$

It is hence clear that Q / Ker (Q on X) $\cong [V \times \underset{n}{\cdots} \times V] \cdot H$ for some integer n.

q.e.d.

II.1.19 Lemma Let p, q and r be prime numbers such that $q \neq p$ and let $t \in \mathbb{N}$ be such that $r \nmid t$ (notice that we allow $t = 1$). Set $T \cong Z_t$ and $Q \cong Z_q$, let V be a faithful, irreducible T-module over GF (r) and let U be a faithful irreducible Q-module over a finite field K of characteristic p. Suppose that $\dim_{GF(r)} V = m$ and $\dim_K U = n$. Let $W = ([U] \cdot Q) \wr_T ([V] \cdot T)$. If B is a subgroup of U Q we denote the image of B in the base group of W by B^*.

(a) U^* may be considered as a faithful irreducible Q^* V T-module over K and $W \cong [U^*] \cdot Q^*$ V T. The K-dimension of U^* is $n.r^m$. If K = GF (p) then G is primitive.

(b) If r = q and M / N is a q-chief factor of W then $W / C_W (M / N)$ is isomorphic to a quotient of T. The chief factors of W above $U^* Q^*$ V are central.

(c) If K = GF (p), L = GF (q) and t = 1 then W has a faithful, irreducible module of dimension a q^b over L, where $a = \deg_p q$ and $b \geqslant 0$.

Proof (a) The proof that U^* is an irreducible K (Q^* V T)-module is an easy adaptation of the proof of the lemma in [23], and

U^* is faithful for $Q^* \, V \, T$ since it is clear that $U^* = C_W(U^*)$. If $K = GF(p)$ then $U^* = C_W(U^*)$ is a minimal normal subgroup of W, and hence W is primitive in this event.

(b) This is obvious in view of II.1.11.

(c) In this case we have $V \cong Z_r$ and $W = U \, Q \wr V$. We notice first of all that there is a KV-submodule U_o of U^* with $U^* / U_o \cong Z_p$ and $[U^*, V] \leq U_o$; this follows from the fact that $U^* \big|_V$ is isomorphic to a product of copies of the regular KV-module.

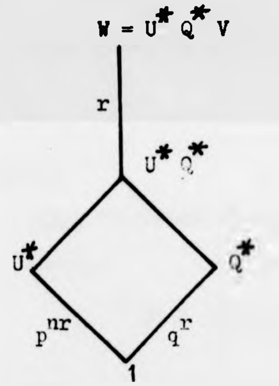

Figure 1.

Such a module certainly has a factor isomorphic to the trivial KV-module. Let $C = C_W(U^* / U_o)$. Clearly, $U^* V \leqslant C$. Since U^* is complemented in W, it is also complemented in C, by J say. Then $U_o J <\cdot U^* J = C$ and $C / U_o J \cong Z_p$. Let Y be an irreducible LC-module with $\mathrm{Ker}\,(C \text{ on } Y) = U_o J$. We observe that $Y \big|_{U^*}$ is irreducible and that $\mathrm{Ker}\,(U^* \text{ on } Y) = U_o$. Consider Y^W. We can certainly find an irreducible submodule X of Y^W such that there is an irreducible LC-submodule Y_o of X with $Y_o \cong Y$. Let H be the homogeneous component of $X \big|_{U^*}$ corresponding to $Y_o \big|_{U^*}$ and let $S = \mathrm{Stab}_W(Y_o \big|_{U^*})$. Then H may be considered as an irreducible LS-module and $X \cong (H_S)^W$. Thus $\dim_L X = |W:S| . \dim_L H$. Since $U_o \lhd S$, it follows that $C \lhd S$. Let H_o be

U^* is faithful for $Q^* V T$ since it is clear that $U^* = C_W(U^*)$. If $K = GF(p)$ then $U^* = C_W(U^*)$ is a minimal normal subgroup of W, and hence W is primitive in this event.

(b) This is obvious in view of II.1.11.

(c) In this case we have $V \cong Z_r$ and $W = U\, Q \wr V$. We notice first of all that there is a KV-submodule U_o of U^* with $U^* / U_o \cong Z_p$ and $[U^*, V] \leq U_o$; this follows from the fact that $U^* \big|_V$ is isomorphic to a product of copies of the regular KV-module.

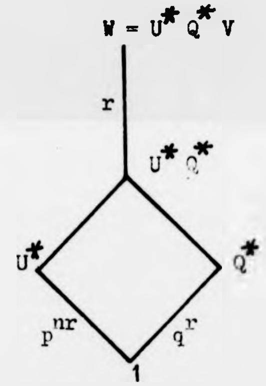

Figure 1.

Such a module certainly has a factor isomorphic to the trivial KV-module. Let $C = C_W(U^* / U_o)$. Clearly, $U^*V \leq C$. Since U^* is complemented in W, it is also complemented in C, by J say. Then $U_oJ \lessdot U^*J = C$ and $C / U_oJ \cong Z_p$. Let Y be an irreducible LC-module with Ker (C on Y) $= U_oJ$. We observe that $Y \big|_{U^*}$ is irreducible and that Ker (U^* on Y) $= U_o$. Consider Y^W. We can certainly find an irreducible submodule X of Y^W such that there is an irreducible LC-submodule Y_o of X with $Y_o \cong Y$. Let H be the homogeneous component of $X \big|_{U^*}$ corresponding to $Y_o \big|_{U^*}$ and let $S = \mathrm{Stab}_W(Y_o \big|_{U^*})$. Then H may be considered as an irreducible LS-module and $X \cong (H_S)^W$. Thus $\dim_L X = |W:S|.\dim_L H$. Since $U_o \trianglelefteq S$, it follows that $C \trianglelefteq S$. Let H_o be the homogeneous component of $H \big|_C$ corresponding to Y_o, and let $S_o = \mathrm{Stab}_S(Y_o)$. Then $\dim_L H = |S:S_o| \dim_L H_o$. Since $V \leqslant C$, we have that $|S:S_o|$ is a power of q. Now, since $U^*V \leqslant C$, it follows that $|S_o:C|$ is a power of q, and so $H_o \big|_C$ is irreducible by I.3.11(a). Hence $H_o \cong Y$. Thus $\dim_L X = |W:S|.|S:S_o| \dim_L Y = q^b.\deg_p q$, where $q^b = |W:S|$.

Finally, if $\{g_1, \ldots, g_{q^b}\}$ is a transversal to S in W, then $\mathrm{Ker}\,(W \text{ on } X) = \bigcap_{i=1}^{q^b} (\mathrm{Ker}\,(S \text{ on } H))^{g_i}$. Since U^* is the unique minimal normal subgroup of W and $U^* \nleqslant \mathrm{Ker}\,(W \text{ on } X)$, we conclude that X is faithful for W. q.e.d.

2. Classes and class maps

<u>II.2.1 Definition</u> By a <u>class</u> of groups we understand a collection $\mathfrak{X}$ of groups with the property:

$$\text{if } H \cong G \in \mathfrak{X} \text{ then } H \in \mathfrak{X}.$$

<u>II.2.2 Notation</u> We shall always use Gothic letters to represent a class of groups. The following will be used as standard notation throughout the text:

$\mathfrak{S}$ denotes the class of all finite soluble groups;

$\mathfrak{N}$ denotes the class of all finite nilpotent groups;

$\mathfrak{A}$ denotes the class of all finite abelian groups;

$\mathfrak{P}$ denotes the class of all primitive soluble groups.

If $\mathfrak{X}$ is a class of groups and π a set of primes, we shall use $\mathfrak{X}_\pi$ to denote the class of π-groups in $\mathfrak{X}$, and if $\pi = \{p\}$ we write $\mathfrak{X}_p$ for $\mathfrak{X}_{\{p\}}$. If n is a positive integer (other than 1) then $\mathfrak{A}(n)$ will denote the class of finite abelian groups of exponent dividing n.

II.2.3 Definitions Let G be a group and $\mathfrak{X}$ a class of groups.

(a) If $G \in \mathfrak{X}$ then G is called an $\mathfrak{X}$-group.

(b) We set $\sigma(G) = \{p : p \text{ is a prime dividing } |G|\}$, and

$$\sigma(\mathfrak{X}) = \bigcup\{\sigma(G) : G \in \mathfrak{X}\}.$$

(c) We define char $(\mathfrak{X}) = \{p \in \mathbb{P} : Z_p \in \mathfrak{X}\}$ and call char $(\mathfrak{X})$ the characteristic of $\mathfrak{X}$.

If $\mathcal{G}$ is a set of groups, we shall use $(G : G \in \mathcal{G})$ or $(\mathcal{G})$ to denote the class generated by $\mathcal{G}$. When $\mathcal{G} = \{G\}$ is a singleton, we shall write simply (G) instead of $(\{G\})$.

II.2.4 Definitions and Notation (a) A class map is a map sending classes of groups to classes of groups.

(b) The product AB of two class maps A and B is defined by composition: thus $AB\mathfrak{X} = A(B\mathfrak{X})$.

(c) We write $A \leq B$ if A and B are class maps such that $A\mathfrak{X} \subseteq B\mathfrak{X}$ for all classes $\mathfrak{X}$ of groups.

(d) If A is a class map, a class $\mathfrak{X}$ is said to be A-closed if $\mathfrak{X} = A\mathfrak{X}$.

(e) The class maps defined in the following list are those that will be of particular importance to us; $\mathfrak{X}$ is a class of groups:

$S\mathfrak{X} = (G : G \leq H \in \mathfrak{X})$;

$Q\mathfrak{X} = (G : G \cong H / N$ for some $H \in \mathfrak{X}$ and $N \trianglelefteq H)$;

$S_n\mathfrak{X} = (G : G \text{ S n } H \in \mathfrak{X})$;

$R_0\mathfrak{X} = (G$: there are normal subgroups $N_1, \ldots, N_r$ of G such that each $G / N_i \in \mathfrak{X}$ and $\bigcap_{i=1}^{r} N_i = 1)$;

$D_0\mathfrak{X} = (G : G$ is a direct product of finitely many $\mathfrak{X}$-groups$)$;

$E_\Phi\mathfrak{X} = (G$: there is a normal subgroup K of G, $K \leq \Phi(G)$, such that $G / K \in \mathfrak{X})$;

$P\mathfrak{X} = (G : Q(G) \cap \mathfrak{P} \subseteq \mathfrak{X})$, the class of groups all of whose primitive epimorphic images lie in $\mathfrak{X}$.

II.2.5 Definitions Let A be a class map.

(a) If $A\mathfrak{X} \supseteq \mathfrak{X}$ for all classes $\mathfrak{X}$, then A is expanding.

(b) If $A\mathfrak{X} = A^2\mathfrak{X}$ for all classes $\mathfrak{X}$, then A is idempotent.

(c) If $A\mathfrak{X} \subseteq A\mathfrak{Y}$ for all classes $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathfrak{X} \subseteq \mathfrak{Y}$, then A is monotonic.

(d) If A is expanding, idempotent and monotonic, then A is called a closure operation.

The following lemma is easily proved from the definitions.

II.2.6 Lemma (a) The class maps S, Q, S_n, R_o, D_o and E_{Φ} are all closure operations.

(b) The class map P is idempotent and monotonic (though not expanding).

(c) A P-closed class is Q-closed. In fact

$$P = Q\,P \text{ and } Q \leq P\,Q.$$

(d) Let π be a set of primes. The classes $\mathfrak{N}_{\pi}$ and $\mathfrak{S}_{\pi}$ are closed under the class maps S, Q, S_n, R_o, D_o, E_{Φ} and P.

(e) Let $n \neq 1$ be a positive integer. Then $\mathfrak{A}(n)$ is closed under S, Q, S_n, R_o and D_o.

II.2.7 Lemma (a) Let $\mathfrak{X}$ be a class. Then $\mathfrak{X} = R_o\mathfrak{X}$ if and only if for all groups G and normal subgroups N_1, N_2 we have:

(α) if G / N_1 and $G / N_2 \in \mathfrak{X}$, then $G / N_1 \cap N_2 \in \mathfrak{X}$.

(b) $R_o \leq S\,D_o$.

Proof (a) Obviously (α) holds when $\mathfrak{X} = R_o\mathfrak{X}$. Suppose conversely that (α) holds. If $N_1, \ldots, N_r$ are normal subgroups of G with $G / N_i \in \mathfrak{X}$ for $1 \leq i \leq r$, then an obvious induction argument shows that $G / \bigcap_{i=1}^{r} N_i \in \mathfrak{X}$. Thus if $\bigcap_{i=1}^{r} N_i = 1$, we have $G \in \mathfrak{X}$, proving that $R_0 \mathfrak{X} \subseteq \mathfrak{X}$ and hence that $\mathfrak{X}$ is R_0-closed.

(b) Let $\mathfrak{X}$ be an $S\ D_0$-closed class and let G be a group with normal subgroups N_1 and N_2 such that $G / N_i \in \mathfrak{X}$ for $i = 1,2$. Consider the homomorphism $\varphi : G \rightarrow G / N_1 \times G / N_2$ with $(g)\varphi = (g N_1, g N_2)$ for $g \in G$. Clearly $\mathrm{Ker}\varphi = N_1 \cap N_2$ and therefore $G / N_1 \cap N_2$ is isomorphic to a subgroup of $G / N_1 \times G / N_2 \in \mathfrak{X}$. Hence $G / N_1 \cap N_2 \in \mathfrak{X}$ and (a) yields the desired consequence that $R_0 \mathfrak{X} \subseteq \mathfrak{X}$.

q.e.d.

There are two particular types of classes with which we are primarily concerned. These we contemplate next.

3. Formations and Schunck classes

II.3.1 Definitions (a) A non-empty class $\mathfrak{F}$ of groups which is both Q-closed and R_0-closed is called a formation. It may be shown that this is equivalent to the condition that $\mathfrak{F} = Q\,R_0 \mathfrak{F}$.

(b) A class $\mathfrak{X}$ of groups such that

$$\emptyset \neq \mathfrak{X} = P\,\mathfrak{X}$$

(where we understand $\emptyset$ to be the empty class) is called a Schunck class.

(c) A class $\mathfrak{H}$ of groups is said to be saturated if $E_{\Phi} \mathfrak{H} = \mathfrak{H}$.

We first concentrate our attention on Schunck classes.

II.3.2 Definition Let $\mathfrak{X}$ be a class of groups and G a group.

(a) A subgroup H of G is called $\mathfrak{X}$-maximal in G if

(i) $H \in \mathfrak{X}$, and

(ii) if $H \leq M \leq G$ with $M \in \mathfrak{X}$, then $H = M$.

(b) A subgroup H of G is called an $\mathfrak{X}$-projector of G if HN/N is $\mathfrak{X}$-maximal in G/N for all $N \trianglelefteq G$.

II.3.3 Examples (a) Let π be a set of primes and consider $\mathfrak{S}_\pi$. The π-maximal subgroups of a group are precisely its Hall π-subgroups. Such subgroups are well-known to have the property described in (b), and so the $\mathfrak{S}_\pi$-projectors of a group also coincide with its Hall π-subgroups. Given a group G we denote the set of all Hall π-subgroups by $\mathrm{Hall}_\pi(G)$. If $\pi = \{p\}$ we often write $\mathrm{Syl}_p(G)$ to denote the set of Sylow p-subgroups of G.

(b) Given a class of groups $\mathfrak{X}$ there need not exist $\mathfrak{X}$-projectors in a given group. Consider $\mathfrak{X} = \mathfrak{A}_2(2)$, the class of elementary abelian 2-groups, and let $G \cong Z_4$. Then G has a unique subgroup N which is an elementary abelian 2-group, namely the normal subgroup of order 2 and index 2 in G. Evidently then, N is $\mathfrak{X}$-maximal in G. However, an $\mathfrak{X}$-maximal subgroup of G/N has order 2. Hence N cannot be an $\mathfrak{X}$-projector of G. On the other hand, if n is a positive integer and π a set of primes, then $\mathfrak{A}_\pi(n)$ is a formation.

Schunck characterised those classes for which projectors exist in all (finite soluble) groups.

II.3.4 Theorem (Schunck [30]) A class $\mathfrak{H}$ is a Schunck class if and only if $\mathfrak{H}$-projectors exist in every group G. If $\mathfrak{H}$ is a Schunck class then the $\mathfrak{H}$-projectors of G form a single characteristic conjugacy class in G.

Proof [19], II . 10, II . 12 and II.17 give the most accessible proof.

q.e.d.

So the theorem of Hall proves that $\mathfrak{S}_\pi^{\cdot}$ is a Schunck class for all sets π of primes. Another example of a Schunck class is $\mathfrak{N}$, the class of nilpotent groups. This fact results from a theorem of Carter [2]. Schunck classes have some convenient intramural properties as well.

<u>II.3.5 Theorem</u> Let $\mathfrak{H}$ be a Schunck class. Then $\mathfrak{H}$ is closed under the class maps P, Q, D_0 and E_Φ.

<u>Proof</u> That $\mathfrak{H}$ is P-closed follows by definition of $\mathfrak{H}$, and the Q-closure follows from II.2.6 (c). We refer to [19], II, 9 (b) for the E_Φ-closure of $\mathfrak{H}$. To show that $\mathfrak{H}$ is D_0-closed it is obviously sufficient to show that the direct product of two $\mathfrak{H}$-groups lies in $\mathfrak{H}$. Let $G_1, G_2 \in \mathfrak{H}$ and let M be a maximal subgroup of $G_1 \times G_2$. If $G_i \not\leq M$, $i = 1,2$, then $G_1 \times G_2 = MG_i \leq N_{G_1 \times G_2}(M \cap G_j)$ and $|G_1 \times G_2 : M| = |G_i : G_i \cap M|$ for $\{i,j\} = \{1,2\}$, and so we deduce that $M \leq G_1 \times G_2$ and that $G_1 \times G_2 / M \cong G_1 / G_1 \cap M \in \mathfrak{H}$. Otherwise we may assume that $G_1 \leq M$, and then we have

$$G_1 \times G_2 / \mathrm{Core}_{G_1 \times G_2}(M) = G_1 \times G_2 / \mathrm{Core}_{G_1 \times G_2}(G_1 \times (M \cap G_2))$$

$$= G_1 \times G_2 / (G_1 \times \mathrm{Core}_{G_2}(M \cap G_2)) \cong G_2 / \mathrm{Core}_{G_2}(M \cap G_2) \in Q(G_2) \subseteq \mathfrak{H},$$

where $\{i,j\} = \{1,2\}$. Clearly, $M \cap G_2 \lessdot G_2$ and so we have shown that $G_1 \times G_2 \in P\mathfrak{H} = \mathfrak{H}$.

q.e.d.

Schunck classes and formations are linked in the following way.

<u>II.3.6 Proposition</u> A class of groups is a saturated formation if and only if it is a Schunck class and a formation.

<u>Proof</u> [19], V. 8.

q.e.d.

It is instructive to see how Schunck classes and saturated formations are characterised by the normal structure of groups contained in them.

II.3.7 Lemma Let $\mathfrak{F}$ be a formation, $G \in \mathfrak{F}$ and A an abelian normal subgroup of G. If $M \trianglelefteq G$ and $A \leq M \leq C_G(A)$, then

$$[A] \cdot G / M \in \mathfrak{F}.$$

Proof [26], VI, 7.21.

q.e.d.

II.3.8 Theorem (a) Let $\mathfrak{H}$ be a Schunck class and $G \in \mathfrak{H}$. If M / N is a complemented chief factor of G, then $[M / N] \cdot A_G(M / N) \in \mathfrak{H}$.

(b) Let $\mathfrak{F}$ be a formation and $G \in \mathfrak{F}$. If M / N is an arbitrary chief factor of G, then $[M / N] \cdot A_G(M / N) \in \mathfrak{F}$.

Proof (a) By II.1.12 and II.3.5 we have

$$[M / N] \cdot A_G(M / N) \in Q(G) \subseteq \mathfrak{H}.$$

(b) Since M / N is an abelian normal subgroup of $G / N \in \mathfrak{F}$, we may apply II.3.7 to yield

$$[M / N] \cdot G / C_G(M / N) \cong [M / N] \cdot (G / N) / C_{G/N}(M / N) \in \mathfrak{F}.$$

q.e.d.

In view of II.1.12 and Definition II.3.1 (b) the property described in (a) of II.3.8 characterises the elements of $\mathfrak{H}$. The same cannot be said of the property described in (b). For with $\mathfrak{F} = \mathfrak{A}(2)$ we see that Z_4 has this property but $Z_4 \notin \mathfrak{F}$. One naturally asks whether the formations which are so characterised can be easily described. The answer is positive and the description is as one would expect in view of II.3.6.

II.3.9 Definition Let f be a map defined on the set of prime numbers such that f (p) is a formation for all primes p. The class $\mathfrak{X}$

is locally defined by f if $\mathfrak{X}$ is the class of groups such that:

$G \in \mathfrak{X}$ if and only if, for each prime p dividing $|G|$

and each p-chief factor M / N of G, $A_G(M / N) \in f(p)$.

The map f is called a local definition for $\mathfrak{X}$.

II.3.10 Theorem (a) (Gaschütz [15]) A locally defined class is a saturated formation.

(b) (Lubeseder [28]) Every saturated formation is locally defined.

Proof A proof of this theorem is contained in [26], VI, Section 7.

q.e.d.

II.3.11 Definition Let $\mathfrak{F}$ be a formation and G a group. Then the $\mathfrak{F}$-residual, $G^{\mathfrak{F}}$, of G is defined to be the intersection of all those normal subgroups N of G with the property that $G / N \in \mathfrak{F}$. Thus G is the uniquely determined smallest normal subgroup of G whose factor group belongs to $\mathfrak{F}$. We say that G is $\mathfrak{F}$-perfect if $G = G^{\mathfrak{F}}$. If π is a set of primes and $G = G^{\mathfrak{S}_\pi}$, then we say that G is π-perfect.

II.3.12 Examples (a) If G is a group then $G^{\mathfrak{A}} = G'$, the derived subgroup of G.

(b) If π is a set of primes then $G^{\mathfrak{S}_\pi} = O^{\pi}(G)$.

II.3.13 Theorem If $\mathfrak{F}$ is a formation, G a group and if φ is a homomorphism defined on G, then

$$(G^{\mathfrak{F}})\varphi = (G\varphi)^{\mathfrak{F}}.$$

Proof [19], V, 13.

q.e.d.

is <u>locally defined</u> by f if $\mathfrak{X}$ is the class of groups such that:

$G \in \mathfrak{X}$ if and only if, for each prime p dividing $|G|$ and each p-chief factor M / N of G, $A_G(M / N) \in f(p)$.

The map f is called a <u>local definition</u> for $\mathfrak{X}$.

<u>II.3.10 Theorem</u> (a) (Gaschütz [15]) A locally defined class is a saturated formation.

(b) (Lubeseder [28]) Every saturated formation is locally defined.

<u>Proof</u> A proof of this theorem is contained in [26], VI, Section 7.
q.e.d.

<u>II.3.11 Definition</u> Let $\mathfrak{F}$ be a formation and G a group. Then the <u>$\mathfrak{F}$-residual</u>, $G^{\mathfrak{F}}$, of G is defined to be the intersection of all those normal subgroups N of G with the property that $G / N \in \mathfrak{F}$. Thus G is the uniquely determined smallest normal subgroup of G whose factor group belongs to $\mathfrak{F}$. We say that G is <u>$\mathfrak{F}$-perfect</u> if $G = G^{\mathfrak{F}}$.

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<u>II.3.13 Theorem</u> If $\mathfrak{F}$ is a formation, G a group and if φ is a homomorphism defined on G, then

$$(G^{\mathfrak{F}})\varphi = (G\varphi)^{\mathfrak{F}}.$$

<u>Proof</u> [19], V, 13.
q.e.d.

II.3.14 Definition Let $\mathfrak{X}$ be a class of groups and $\mathfrak{Y}$ a formation. We set

$$\mathfrak{X}\mathfrak{Y} = (G : G \text{ a group, } G^{\mathfrak{Y}} \in \mathfrak{X}),$$

the formation product of $\mathfrak{X}$ with $\mathfrak{Y}$.

II.3.15 Theorem Let $\mathfrak{X}$ and $\mathfrak{Y}$ be formations. Then $\mathfrak{X}\mathfrak{Y}$ is a formation.

Proof [19], VII, 6.
q.e.d.

Local definitions were extensively investigated by Carter and Hawkes [4]. Amongst the more important ideas introduced by them is the notion of an integrated full local definition.

II.3.16 Definitions (Carter and Hawkes [4]). Let f be a local definition of the saturated formation $\mathfrak{F}$.

(a) The local definition f is said to be integrated if f (p) $\subseteq \mathfrak{F}$ for all primes p.

(b) We say that f is full if $\mathfrak{S}_p$ f (p) = f (p) for all primes p.

II.3.17 Theorem (Carter and Hawkes [4]) Let $\mathfrak{F}$ be a saturated formation.

(a) There is an integrated local definition of $\mathfrak{F}$.

(b) If f_1 and f_2 are integrated local definitions of $\mathfrak{F}$ then

$$\mathfrak{S}_p \, f_1 (p) = \mathfrak{S}_p \, f_2 (p).$$

Proof [4] Lemma 2.1 and Theorem 2.2.

II.3.18 Lemma If $\mathfrak{F}$ is a saturated formation and p $\in$ char ($\mathfrak{F}$) then $\mathfrak{S} \subseteq \mathfrak{F}$.

Proof Since $Z_p \in \mathfrak{F}$, it is clear that $\alpha(p) \in \mathfrak{F}$. Let P be a p-group. Since $P / \Phi (P)$ is an elementary abelian p-group, we have

$$P \in E_{\Phi} \mathfrak{F} = \mathfrak{F}.$$

q.e.d.

It is now a straightforward corollary of II.3.17 and II.3.18 to prove that every saturated formation possesses a unique integrated full local definition.

<u>II.3.19 Corollary</u> If $\mathfrak{F}$ is a saturated formation then $\mathfrak{F}$ possesses a unique full and integrated local definition.

<u>Proof</u> By II.3.17 there is an integrated definition g of $\mathfrak{F}$. Define f by

$$f(p) = \mathfrak{S}_p\, g(p)$$

for all primes p. Notice that if $p \notin$ char $(\mathfrak{F})$ then $g(p) = \emptyset$ and hence $f(p) = \emptyset$. Let $\mathfrak{F}(f)$ be the saturated formation locally defined by f. Clearly $g(p) \subseteq f(p)$ and so $\mathfrak{F} \subseteq \mathfrak{F}(f)$. Let $G \in \mathfrak{F}(f)$ and let M / N be a p-chief factor of G. Then $A_G(M / N) \in f(p) = \mathfrak{S}_p\, g(p)$. Since $A_G(M / N)$ is faithfully and irreducibly represented on M / N over GF (p), it follows that $O_p(A_G(M / N)) = 1$ and so $A_G(M / N) \in g(p)$. Therefore $G \in \mathfrak{F}$ and we have $\mathfrak{F} = \mathfrak{F}(f)$. It remains to check that $f(p) \subseteq \mathfrak{F}$. Let $H \in f(p)$ and let M / N be a p-chief factor of H with $M \leq H^{g(p)} \leq O_p(H)$. Then, since $O_p(A_H(M / N)) = 1$, we have

$$A_H(M / N) \in Q(H / O_p(H)) \subseteq g(p).$$

As $H / H^{g(p)} \in g(p) \subseteq \mathfrak{F}$, it follows that $H \in \mathfrak{F}$. The uniqueness of f follows immediately from II.3.17 (b).

q.e.d.

There is another result related to II.3.18.

<u>II.3.20 Lemma</u> If $\mathfrak{F}$ is a saturated formation then char $(\mathfrak{F}) = \sigma(\mathfrak{F})$.

In particular, if $G \in \mathfrak{F}$ and $p \mid |G|$ then $Z_p \in \mathfrak{F}$.

Proof Obviously char $(\mathfrak{X}) \subseteq \sigma(\mathfrak{X})$ for any class $\mathfrak{X}$. Let f be a local definition of $\mathfrak{F}$ and let $p \in \sigma(\mathfrak{F})$. Then there exists a group $G \in \mathfrak{F}$ with a p-chief factor M / N. Since $A_G (M / N) \in f(p)$ it follows that $f(p) \neq \emptyset$ and hence, by Q-closure of $f(p)$, that $1 \in f(p)$. It now follows at once that $Z_p \in \mathfrak{F}$.
q.e.d.

II.3.21 Lemma Let $\mathfrak{F}$ be a saturated formation with local definition f and let $G \in f(p) \cap \mathfrak{F}$. Then $[V] \cdot G \in \mathfrak{F}$ whenever V is a GF (p) G-module.

Proof Let V be a GF (p) G-module and let M / N be a chief factor of a chief series of $[V] \cdot G$ for G running through V. Let q be the prime divisor of $|M / N|$. If $V \leq N$ then there is a chief factor M_o / N_o of G such that $V G / C_{VG}(M / N) \cong G / C_G (M_o / N_o)$. Since $G \in \mathfrak{F}$ we have $A_G (M_o / N_o) \cong A_{VG} (M / N) \in f(q)$. If $M \leq V$ then $A_{VG} (M / N) \in Q(G) \subseteq f(p) \cap \mathfrak{F}$. Hence $V G \in \mathfrak{F}$.
q.e.d.

II.3.22 Corollary If f is an integrated definition of $\mathfrak{F}$ and $G \in f(p)$ then $[V] \cdot G \in \mathfrak{F}$ for all GF (p) G-modules V.

II.3.23 Theorem (Doerk [9], 2.2) Let f be a local definition of the saturated formation $\mathfrak{F}$.

(a) If f (p) is S_n-closed (S-closed) for all primes p then $\mathfrak{F}$ is S_n-closed (respectively, S-closed).

(b) If f is full and integrated and $\mathfrak{F}$ is S_n-closed (S-closed) then f (p) is S_n-closed (respectively, S-closed) for all primes p.

There is a useful local specialisation of II.3.23 (a).

II.3.24 Lemma Let $\mathfrak{F}$ be a saturated formation with local definition f and let $G = V\,H$ be a primitive group with socle V and $V \cap H = 1$ such that $S_n(G) \subseteq \mathfrak{F}$. Then $S_n(H) \subseteq f(p)$, where p is the prime divisor of $|V|$.

Proof Certainly $H \in f(p)$. Let $N \trianglelefteq H$ and consider $V\big|_N$. By Clifford's Theorem, we may decompose $V\big|_N$ into a direct sum of irreducible GF (p) N-modules, viz.

$$V\big|_N = V_1 \oplus \dots \oplus V_r.$$

Let $C_i = \mathrm{Ker}\,(N \text{ on } V_i)$. The V_i's may be considered as chief factors of $V\,N$. Since $V\,N \trianglelefteq G$ we have $V\,N \in \mathfrak{F}$ and so $N / C_i \in f(p)$. But V is faithful for N and hence $\bigcap_{i=1}^{r} C_i = 1$, whence $N \in R_0\, f(p) = f(p)$.
q.e.d.

4. Ranking functions and arithmetically defined Schunck classes

II.4.1 Definition (a) We denote the set of all primes by $\mathbb{P}$ and the set of all sets of positive integers by $\mathcal{P}(\mathbb{N})$. A map $\mathfrak{R} : \mathbb{P} \longrightarrow \mathcal{P}(\mathbb{N})$ will be called a ranking function. The image of a prime p under $\mathfrak{R}$ will be written $\mathfrak{R}_p$. If $\mathfrak{R}_p \neq \emptyset$ for all $p \in \mathbb{P}$, then we say that $\mathfrak{R}$ is a full ranking function.

(b) Let G be a group and let M / N be a chief factor of G. If p is the prime divisor of $|M / N|$ then M / N may be considered as an irreducible GF (p) G-module. We set

$$r\,(M / N) = \dim_{GF(p)} M / N,$$

the rank of M / N. The dimension of an absolutely irreducible component of M / N, considered as a GF (p) G-module, is denoted by $r_a(M / N)$ (by I.2.10 (b) and I.2.15 this is independent of choice of absolutely irreducible component). We will call $r_a(M / N)$ the absolute rank of M / N. Often, a module will be considered as a chief factor of a semidirect product or, alternatively, a chief factor of a group will be considered as a module for that group. We will interchange the terms rank and dimension appropriately in an attempt to remain consistent with their common usage, and to indicate in which sense the chief factor or module is being thought of.

(c) Let $\mathfrak{R}$ be a ranking function. Then we define the classes $\mathfrak{J}(\mathfrak{R})$ and $\mathfrak{J}_a(\mathfrak{R})$ as follows:

$$\mathfrak{J}(\mathfrak{R}) = (G \in \mathfrak{G} : \text{for all } p \in \mathbb{P} \text{ and all p-chief factors } M/N \text{ of } G,\ r(M/N) \in \mathfrak{R}_p),$$

$$\mathfrak{J}_a(\mathfrak{R}) = (G \in \mathfrak{G} : \text{for all } p \in \mathbb{P} \text{ and all p-chief factors } M/N \text{ of } G,\ r_a(M/N) \in \mathfrak{R}_p).$$

II.4.2 Examples (a) Let $\mathfrak{R}_p = \mathbb{N}$ for all primes p. Then $\mathfrak{J}(\mathfrak{R}) = \mathfrak{J}_a(\mathfrak{R}) = \mathfrak{G}$.

(b) Let $\mathfrak{S}_p = \{1\}$ for all primes p.. Then $\mathfrak{J}(\mathfrak{S}) = \mathfrak{U}$, the class of all supersoluble groups.

(c) Let $\pi \subseteq \mathbb{P}$. If $\mathfrak{R}$ is the ranking function defined by

$$\mathfrak{R}_p = \mathbb{N} \text{ for all } p \in \pi \text{, and}$$
$$\mathfrak{R}_p = \emptyset \text{ otherwise,}$$

then clearly $\mathfrak{J}(\mathfrak{R}) = \mathfrak{J}_a(\mathfrak{R}) = \mathfrak{G}_\pi$.

Define $\mathfrak{S} : \mathbb{P} \to \mathcal{P}(\mathbb{N})$ by

$$\mathfrak{S}_p = \{1\} \text{ for all } p \in \pi \text{, and}$$
$$\mathfrak{S}_p = \emptyset \quad \text{otherwise.}$$

Then $\mathfrak{J}(\mathfrak{S}) = \mathfrak{U} \cap \mathfrak{G}_\pi$, the class of supersoluble π-groups. In general $\mathfrak{J}(\mathfrak{S}) \neq \mathfrak{J}_a(\mathfrak{S})$. Suppose that $\{2, 3\} \subseteq \pi$ and consider G = Alt (4), the alternating group of degree 4. The group G is primitive and has unique chief series

$$1 < N < G$$

in which $|N| = 4$ and $|G : N| = 3$. It is easy to check (using I.1.7) that $G \in \mathfrak{F}_a(\mathfrak{A})$, although G is certainly not supersoluble.

(d) Let $\pi \subseteq \mathbb{P}$ and define $\mathfrak{R}$ by

$$\mathfrak{R}_p = \{1\} \quad \text{for } p \in \pi, \text{ and}$$
$$\mathfrak{R}_p = \mathbb{N} \quad \text{otherwise.}$$

Then $\mathfrak{F}(\mathfrak{R})$ is the class of π-supersoluble groups, which we shall denote $\mathfrak{U}(\pi)$.

(e) (Huppert [26], VI, 8.3) Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$.

Define $\mathfrak{R} : \mathbb{P} \longrightarrow \mathcal{P}(\mathbb{N})$ by

$$\mathfrak{R}_q = \emptyset \text{ if } q \mid n;$$
$$\mathfrak{R}_p = \{m \in \mathbb{N} : m \mid n\} \text{ if } p \nmid n; \text{ and}$$
$$\mathfrak{R}_q = \mathbb{N} \text{ for all other } q.$$

Then $\mathfrak{F}(\mathfrak{R})$ is a saturated formation with local definition f defined by

$$f(q) = \emptyset \text{ if } q \mid n;$$
$$f(p) = \mathfrak{A}(p^n - 1) \text{ if } p \nmid n; \text{ and}$$
$$f(q) = \mathfrak{S} \text{ otherwise.}$$

One of our aims in Chapter III is to characterise all those ranking functions $\mathfrak{R}$ such that $\mathfrak{F}(\mathfrak{R})$ is a saturated formation. All those classes introduced in II.4.2 are in fact saturated formations. This is not always the case, as Example II.4.3 will show:

<u>II.4.3 Example</u> Let $\mathfrak{R} : \mathbb{P} \longrightarrow \mathcal{P}(\mathbb{N})$ be defined by

$$\mathfrak{R}_2 = \{2\},$$
$$\mathfrak{R}_p = \mathbb{N} \text{ for all odd primes } p.$$

Let $G = SL(2, 3)$. The group G has a unique chief series:

$$1 < \Phi(G) < F(G) < G$$

in which $\Phi(G)$ is the unique minimal normal subgroup of G and is of order 2, the factor $F(G) / \Phi(G)$ is of order 4, and hence of rank 2, whilst $G / F(G) \cong Z_3$. Thus $G / \Phi(G)$ has a 2-chief factor of rank 2 and a 3-chief factor of rank 1, and so $G / \Phi(G) \in \mathfrak{J}(\mathfrak{R})$. However, $\Phi(G)$ is a 2-chief factor of G of rank 1, and $G \notin \mathfrak{J}(\mathfrak{R})$. Thus $\mathfrak{J}(\mathfrak{R})$ is not saturated.

We observe also for this $\mathfrak{R}$ that $\mathfrak{J}_a(\mathfrak{R}) = \mathfrak{S}_2$. For suppose that $H \in \mathfrak{J}_a(\mathfrak{R})$ and that H has a non-trivial 2-chief factor, S / T say. The absolute rank of S / T must be 2, and so appealing to I.4.3, the order of $H / C_H(S / T)$ is divisible by 2, and so there is a 2-chief factor of H above $C_H(S / T)$. In particular, $H = O^2(H)$ and so $O^{2'}(H) < H$. Let $Q = O^{2'}(H)$ and let Q / B be a chief factor of H ($Q \neq 1$ since $2 \mid |H|$). But $Q \leq C_H(Q / B)$, so $A_H(Q / B)$ is a $2'$-group. This is a contradiction, as Q / B is a 2-chief factor. It therefore follows that $\mathfrak{J}_a(\mathfrak{R}) \subseteq \mathfrak{S}_2$. Equality follows easily.

Example II.4.3 has introduced us to a technical problem. Given a ranking funtion $\mathfrak{R}$ there may exist a different ranking function $\mathfrak{R}'$ such that $\mathfrak{J}_a(\mathfrak{R}) = \mathfrak{J}_a(\mathfrak{R}')$. Indeed, such functions exist such that $\mathfrak{J}(\mathfrak{R}) = \mathfrak{J}(\mathfrak{R}')$. In the example of II.4.2 (e) take

$$\mathfrak{R}_2 = \mathfrak{R}_3 = \emptyset;$$
$$\mathfrak{R}_5 = \{1, 2, 3, 6\}; \text{ and}$$
$$\mathfrak{R}_q = \mathbb{N} \text{ for all primes } q \geq 7.$$

Suppose that $H \in \mathfrak{J}(\mathfrak{R})$ and that H possesses a 5-chief factor M / N of H of rank 2. Since $\mathfrak{R}_2 = \emptyset$, the group of automorphisms induced on M / N is of $2'$-order, and so by [26], VI, 8.1, $A_H(M / N)$ is cyclic of order dividing $5^2 - 1 = 24$. However, $\mathfrak{R}_2 = \mathfrak{R}_3 = \emptyset$ and so $A_H(M / N)$ is trivial, since $A_H(M / N) \in Q(H) \subseteq \mathfrak{F}$, contradicting the assumption that $r(M / N) = 2$. This shows that if we take

$$\mathfrak{R}'_2 = \mathfrak{R}'_3 = \emptyset;$$

$$\mathfrak{R}'_5 = \{1, 3, 6\};$$

$$\mathfrak{R}'_q = \mathbb{N} \text{ for all } q \geqslant 7.$$

Then $\mathfrak{R} \neq \mathfrak{R}'$ but $\mathfrak{F}(\mathfrak{R}) = \mathfrak{F}(\mathfrak{R}')$.

To avoid this problem we look for a "canonical" ranking function associated with a given ranking function.

<u>II.4.4 Definition</u> Let $\mathfrak{R}$ be a ranking function. Then $\min\mathfrak{R}$ and $\min_a\mathfrak{R}$, the <u>minimum</u> and <u>absolute minimum</u> respectively of $\mathfrak{R}$, are ranking functions defined for all $p \in \mathbb{P}$ as follows:

$$\min\mathfrak{R}_p = \{n \in \mathbb{N} : \text{there exists } G \in \mathfrak{F}(\mathfrak{R}) \text{ and p-chief factor } M / N \text{ of } G \text{ such that } r(M / N) = n\},$$

$$\min_a\mathfrak{R}_p = \{n \in \mathbb{N} : \text{there exists } G \in \mathfrak{F}_a(\mathfrak{R}) \text{ and p-chief factor } M / N \text{ of } G \text{ such that } r_a(M / N) = n\}.$$

We call $\mathfrak{R}$ <u>minimal</u> (<u>absolute minimal</u>) if $\mathfrak{R} = \min\mathfrak{R}$ (respectively, $\mathfrak{R} = \min_a\mathfrak{R}$).

We see that $\min\mathfrak{R}$ and $\min_a\mathfrak{R}$ have the desired properties.

<u>II.4.5 Lemma</u> Let $\mathfrak{R}$ be a ranking function.

(a) $\mathfrak{F}(\min \mathfrak{R}) = \mathfrak{F}(\mathfrak{R})$;

$\mathfrak{F}_a(\min_a \mathfrak{R}) = \mathfrak{F}_a(\mathfrak{R})$.

(b) $\left.\begin{array}{l} \min \mathfrak{R}_p \subseteq \mathfrak{R}_p \\ \min_a \mathfrak{R}_p \subseteq \mathfrak{R}_p \end{array}\right\}$ for all $p \in \mathbb{P}$.

(c) If $\mathfrak{R}'$ is another ranking function and $\mathfrak{F}(\mathfrak{R}) = \mathfrak{F}(\mathfrak{R}')$ ($\mathfrak{F}_a(\mathfrak{R}) = \mathfrak{F}_a(\mathfrak{R}')$), then $\min \mathfrak{R} = \min \mathfrak{R}'$ (respectively, $\min_a \mathfrak{R} = \min_a \mathfrak{R}'$).

(d) For all $p \in \mathbb{P}$,

$$\min \mathfrak{R}_p = \bigcap \mathfrak{R}'_p, \text{ and}$$
$$\min_a \mathfrak{R}_p = \bigcap \mathfrak{R}'_p$$

where in the former case the intersection is taken over all ranking functions $\mathfrak{R}'$ with $\mathfrak{F}(\mathfrak{R}) = \mathfrak{F}(\mathfrak{R}')$ and in the latter case over all the ranking functions $\mathfrak{R}'$ with $\mathfrak{F}_a(\mathfrak{R}) = \mathfrak{F}_a(\mathfrak{R}')$.

Proof (a), (b), and (c) are clear from the definitions. (d) follows from (b) and (c).

q.e.d.

Because of II.4.5 we may henceforth assume that each ranking function we consider is either minimal or absolute minimal, depending on the context, unless otherwise stated.

In order to simplify language, we introduce the following terms.

II.4.6 Definitions (a) Let $\mathfrak{R}$ be a ranking function and let G be a group. We say that the p-chief factor M / N of G is $\mathfrak{R}$-admissible if $r(M / N) \in \mathfrak{R}_p$, and $\mathfrak{R}_a$-admissible if $r_a(M / N) \in \mathfrak{R}_p$.

(b) A formation $\mathfrak{F}$ is said to be ranked by $\mathfrak{R}$ if $\mathfrak{F} = \mathfrak{F}(\mathfrak{R})$ and absolutely ranked by $\mathfrak{R}$ if $\mathfrak{F} = \mathfrak{F}_a(\mathfrak{R})$ for some ranking function $\mathfrak{R}$.

<u>II.4.7 Lemma</u> Let $\mathfrak{F}$ be a formation ranked by $\mathfrak{R}$. Then $G \in \mathfrak{F}$ if and only if every chief factor of G is $\mathfrak{R}$-admissible. Similarly, if $\mathfrak{F}$ is absolutely ranked by $\mathfrak{R}$ then $G \in \mathfrak{F}$ if and only if every chief factor of G is $\mathfrak{R}_a$-admissible.

<u>Proof</u> Immediate by definition.

q.e.d.

We observed in Example II.4.3 that given the ranking function $\mathfrak{R}$ then $\mathfrak{F}(\mathfrak{R})$ need not be saturated. However:

<u>II.4.8 Theorem</u> Let $\mathfrak{R}$ be a ranking function. Then $\mathfrak{F}(\mathfrak{R})$ and $\mathfrak{F}_a(\mathfrak{R})$ are both formations.

<u>Proof</u> We shall show that $\mathfrak{F}_a(\mathfrak{R})$ is a formation. That $\mathfrak{F}(\mathfrak{R})$ is a formation is proved similarly.

<u>$\mathfrak{F}_a(\mathfrak{R})$ is Q-closed</u>: Let $G \in \mathfrak{F}(\mathfrak{R})$ and $N \trianglelefteq G$. Consider a chief series of G running through N and let S / T be a chief factor with $N \leq T$. Then S / T is an irreducible G / N-module, and it is clear that $(S / N) / (T / N)$ is a chief factor of G / N such that $S / T \cong (S / N) / (T / N)$ as G / N-modules, and $r_a(S / T) = r_a((S / N) / (T / N))$. Since each chief factor of G / N appears in this manner, it follows that each chief factor of G / N is $\mathfrak{R}_a$-admissible. Hence $G / N \in \mathfrak{F}_a(\mathfrak{R})$.

<u>$\mathfrak{F}_a(\mathfrak{R})$ is R_0-closed</u>: By II.2.7 it suffices to show that if N_1 and N_2 are normal subgroups of a group G such that $(G / N_1, G / N_2) \subseteq \mathfrak{F}_a(\mathfrak{R})$ then $G / N_1 \cap N_2 \in \mathfrak{F}_a(\mathfrak{R})$. We consider two chief series of G running through $N_1 \cap N_2$ and $N_1 N_2$ one passing through N_1 and the other passing through N_2. Since $G / N_1 \in \mathfrak{F}_a(\mathfrak{R})$ we see that the chief factors of G above N_1 are $\mathfrak{R}_a$-admissible. Similarly, the chief factors of G above N_2 are $\mathfrak{R}_a$-admissible. Considered as G-groups, we have the isomorphism

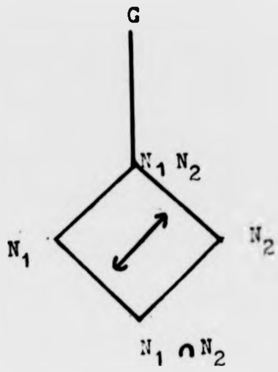

Figure 2.

$$N_1 / N_1 \cap N_2 \cong N_1 N_2 / N_2$$

and so the chief factors of G between $N_1 \cap N_2$ and N_1 are in one-to-one correspondence with the chief factors of G between N_2 and $N_1 N_2$, corresponding chief factors being G-isomorphic, and hence of the same absolute rank. Thus we deduce that those chief factors of G between $N_1 \cap N_2$ and N_1 are $\mathfrak{R}_a$-admissible, and therefore all chief factors of G above $N_1 \cap N_2$ are $\mathfrak{R}_a$-admissible.
q.e.d.

It is therefore a natural question to ask when a ranked formation is saturated. We are also motivated by quite different reasons. In recent work, Gaschütz [18] has investigated the existence and conjugacy of the so-called generalised Sylow subgroups. This notion has been generalised by Hawkes [24].

II.4.9 Definition (a) We denote the set of prime powers by $\mathbb{P}^*$. If $\Omega \subseteq \mathbb{P}^*$, then we set $\Omega' = \mathbb{N} \setminus \Omega$.

(b) Let G be a group and $\Omega \subseteq \mathbb{P}^*$. We call G an <u>$\Omega$-group</u> if $|G : U| \in \Omega$ whenever $U <\cdot G$.

(c) (Gaschütz [18]) Let G be a group and $\Omega \subseteq \mathbb{P}^*$. A subgroup S of G is a <u>Sylow Ω-subgroup</u> of G if S is an Ω-group and whenever $S \leqslant K \leqslant H \leqslant G$ then $|H : K| \in \Omega'$.

(d) (Hawkes [24]) Let G be a group and $\Omega \subseteq \mathbb{P}^*$. A subgroup S of G is a <u>Gaschütz Ω-subgroup</u> of G if S is an Ω-group and whenever $S \leqslant K <\cdot H \leqslant G$ then $|H : K| \in \Omega'$.

<u>II.4.10 Example</u> Let $\pi \subseteq \mathbb{P}$ and set Ω to be the set of all powers of primes in π. Then the Sylow Ω-subgroups of a group G coincide with its Gaschütz Ω-subgroups. They are precisely the Hall π-subgroups of G.

Since Sylow Ω-subgroups are Gaschütz Ω-subgroups, it is the latter concept which is of most interest. In [24], Hawkes shows that Gaschütz Ω-subgroups, if they exist in a group, are projectors for certain Schunck classes, the general form of which is described in the next definition.

<u>II.4.11 Definition</u> (a) Let G be a primitive group with socle N. We set $\partial G = |N|$.

(b) Let $\Omega \subseteq \mathbb{P}^*$. We write

$$\mathcal{P}_\Omega = (G \in \mathcal{P} : \partial G \in \Omega).$$

It follows from II.2.6 (b) that $P\mathcal{P}_\Omega$ is P-closed, and so is a Schunck class. We call $P\mathcal{P}_\Omega$ a <u>Gaschütz class</u>.

<u>II.4.12 Lemma</u> Let $\Omega \subseteq \mathbb{P}^*$. Then

$$P\mathcal{P}_\Omega = (G : \text{if } M <\cdot G \text{ then } |G : M| \in \Omega)$$

<u>Proof</u> Set $\mathcal{J} = (G : \text{if } M <\cdot G \text{ then } |G : M| \in \Omega)$.

Let $G \in P\mathfrak{P}_\Omega$. If $N \leqslant G$ such that $G / N \in \mathfrak{P}$ then it follows fom II.1.9 that there is a maximal subgroup H of G such that $N = \mathrm{core}_G(H)$. Clearly H / N complements the socle of G / N and so $\partial(G / N) = |G / N : H / N| = |G : H| \in \Omega$. Since every maximal subgroup of G is associated with a primitive quotient of G in this manner (again by II.1.9) we see that $G \in \mathfrak{h}_\sigma$, and for the same reason, $\mathfrak{h}_\sigma \subseteq P\mathfrak{P}_\Omega$.

q.e.d.

Although a $P\mathfrak{P}_\Omega$-projector of a group G is not necessarily a Gaschütz Ω-subgroup of G - indeed a group G need not possess such subgroups for a particular choice of Ω - Hawkes is able to characterise the Gaschütz classes $P\mathfrak{P}_\Omega$ for which the projectors in G are always the Gaschütz Ω-subgroups of G. We are not concerned with the existence of Gaschütz Ω-subgroups in this thesis but rather with the Gaschütz classes. In particular, a central question is: for what sets $\Omega \subseteq \mathbb{P}^*$ is $P\mathfrak{P}_\Omega$ a saturated formation? The totality of such classes turns out to be precisely the ranked saturated formations. Before proving this statement formally, we prove a useful elementary lemma.

<u>II.4.13 Lemma</u> Let $\mathfrak{X}$ be a Q-closed class and let $\mathfrak{Y}$ be a formation. If G is of minimal order in $\mathfrak{X} \setminus \mathfrak{Y}$, then G has a unique minimal normal subgroup. If $\mathfrak{Y}$ is in addition saturated, then G is primitive.

<u>Proof</u> Let N_1 and N_2 be minimal normal subgroups of G and suppose that $N_1 \cap N_2 = 1$. Since $\mathfrak{X}$ is Q-closed, the minimal choice of G forces $G / N_i \in \mathfrak{Y}$ for $i = 1, 2$. Hence, by II.2.7 (a), $G = G / N_1 \cap N_2 \in \mathfrak{Y}$, contrary to the choice of G. We therefore deduce that G has a unique minimal normal subgroup, N say. Suppose

that $\mathfrak{Y}$ is also saturated. Since $G / N \in \mathfrak{Y}$ we must have $N \cap \Phi(G) = 1$, and so by II.1.2 (b) there is a complement H of N in G. The uniqueness of N forces $\mathrm{Core}_G(H) = 1$, and so by II.1.9, G is primitive.

<u>II.4.14 Proposition</u> The following statements are equivalent for a class $\mathfrak{H}$.

(<u>a</u>) $\mathfrak{H}$ is a Gaschütz class and a formation.

(<u>b</u>) $\mathfrak{H}$ is a ranked saturated formation.

<u>Proof</u> <u>(a)$\Rightarrow$(b)</u>. Let $\Omega \subseteq \mathbb{P}^*$ be such that $\mathfrak{H} = P\mathfrak{F}_\Omega$, and define a ranking function $\mathfrak{R}$ by

$$\mathfrak{R}_p = \{ n \in \mathbb{N} : p^n \in \Omega \}$$

for each $p \in \mathbb{P}$. If $G \in \mathfrak{J}(\mathfrak{R})$ and $H <\!\cdot\, G$ then by II.1.9 and II.1.12, $|G : H|$ is the order of a complemented q-chief factor S / T of G for some $q \in \mathbb{P}$. Since $r(S / T) \in \mathfrak{R}_q$ we have

$$|G : H| = |S : T| \in \Omega .$$

By II.4.12 therefore, $G \in P\mathfrak{F}_\Omega$ and we have

$$\mathfrak{J}(\mathfrak{R}) \subseteq P\mathfrak{F}_\Omega .$$

Suppose, on the other, that G is of minimal order in $P\mathfrak{F}_\Omega \setminus \mathfrak{J}(\mathfrak{R})$. By II.4.13, the group G has a unique minimal normal subgroup, N say. Since $P\mathfrak{F}_\Omega$ is a formation, it follows from II.3.7 that $[N] \cdot A_G(N) \in P\mathfrak{F}_\Omega$. Hence, if $C_G(N) > N$ then $[N] \cdot A_G(N) \in \mathfrak{J}(\mathfrak{R})$, and $r(N)$ must lie in $\mathfrak{R}_p$, where p is the prime divisor of $|N|$. Since $G / N \in \mathfrak{J}(\mathfrak{R})$, it follows that every chief factor of G is $\mathfrak{R}$-admissible, and hence that $G \in \mathfrak{J}(\mathfrak{R})$. Since this is against the choice of G, our last supposition must have been false, forcing $N = C_G(N)$. Now G is primitive. Let $H <\!\cdot\, G$ with $N.H = G$. Then $|N| = |G : H| \in \Omega$ by II.4.12. But again this implies that N is $\mathfrak{R}$-admissible as a chief factor of G, implying that $G \in \mathfrak{F}(\mathfrak{R})$. This contradicts the choice of G, proving the result in this direction.

<u>(b)$\Rightarrow$(a)</u>. Let $\mathfrak{R}$ be the (minimal) ranking function such that $\mathfrak{G} = \mathfrak{F}(\mathfrak{R})$, and define $\Omega \subseteq \mathbb{P}^{\#}$ by

$$\Omega = \left\{ p^n : p \in \mathbb{P} \text{ and } n \in \mathfrak{R}_p \neq \emptyset \right\}.$$

The argument now runs just as in (a) to prove that

$P\mathfrak{F}_{\Omega} = \mathfrak{F}(\mathfrak{R})$.

q.e.d.

Thus to characterise the saturated ranking functions is to characterise the Gaschütz classes which are formations. We shall see that we simultaneously characterise those Gaschütz classes which are subgroup closed.

Chapter III

Chapter III. Ranked saturated formations.

It is convenient to divide the characterisation of ranked saturated formations into two parts. In the first part we investigate the behaviour of ranking functions defining saturated formations and the second part will contain the characterisation theorems.

1. Ranking functions for saturated formations

In this section $\mathfrak{R}$ will always be a ranking function with $\mathfrak{F}(\mathfrak{R})$ saturated and $\mathfrak{S}$ will be an arbitrary ranking function. Recalling the general hypothesis in Section 4 of Chapter II, we assume that $\mathfrak{R}$ and $\mathfrak{S}$ are minimal. By II.3.19 we can choose f to be the unique full, integrated local definition of $\mathfrak{F}(\mathfrak{R})$. Our first result about $\mathfrak{R}$ is extremely easy to prove.

III.1.1 Lemma For all primes p,

$$\mathfrak{R}_p \neq \emptyset \text{ if and only if } 1 \in \mathfrak{R}_p.$$

Proof That $1 \in \mathfrak{R}_p$ implies that $\mathfrak{R}_p \neq \emptyset$ is trivial. If $\mathfrak{R}_p \neq \emptyset$ there is a group $G \in \mathfrak{F}(\mathfrak{R})$ such that $p \mid |G|$ (here we use the minimality of $\mathfrak{R}$). By II.3.20 we have $Z_p \in \mathfrak{F}(\mathfrak{R})$ and hence $1 \in \mathfrak{R}_p$. q.e.d.

Further progress is not possible without the next result. The succinctness of its statement belies the complexity of its proof!

III.1.2 Theorem $\mathfrak{F}(\mathfrak{R})$ is S_n-closed. In particular, $f(p)$ is S_n-closed for all $p \in \mathbb{P}$.

Proof We prove that $f(p)$ is S_n-closed for all primes p, and apply II.3.23 (a). To this end we choose a group G of minimal order subject to the following condition:

($*$) : $G \in f(p)$ for some prime p and there is a subnormal subgroup of G not belonging to $f(p)$.

We aim for a contradiction.

The minimality of G with property ($*$) implies that there is a maximal normal subgroup M of G with $M \notin f(p)$. Since every subgroup of an abelian group is isomorphic to a quotient of that group, we can assume that

(1) G is not abelian.

Let N be a minimal normal subgroup of G, and let t be the prime divisor of $|N|$. If $N \not\leq M$ then $NM = G$ and $N \cap M = 1$ by II.1.7, yielding

$$M \cong NM/N = G/N \in f(p),$$

a contradiction. Hence $N \leq M$. Suppose N_1 is also a minimal normal subgroup of G with $N_1 \cap N = 1$. We must again have $N_1 \leq M$. Now, $M/N \trianglelefteq G/N \in f(p)$, and so the minimal choice of G gives

$$M/N \in f(p)$$

and similarly, $M/N_1 \in f(p)$. But now we have

$$M \cong M/N_1 \cap N \in R_0\, f(p) = f(p),$$

a contradiction. We have therefore shown that

(2) N is the unique minimal normal subgroup of G and $M/N \in f(p)$.

We wish to show that $M \in \mathfrak{Z}(\mathfrak{X})$. Since $M/N \in f(p) \subseteq \mathfrak{Z}(\mathfrak{X})$, it is sufficient to prove that $M/C_M(N) \in f(t)$; for if R/S is a chief factor of M below N (i.e. if $R \leq N$) then we have $M/C_M(R/S) \in Q(M/C_M(N)) \subseteq f(t)$ since N is abelian. This will show that the chief factors of M are all $\mathfrak{X}$-admissible, and hence $M \in \mathfrak{Z}(\mathfrak{X})$. Now, because N may be considered as a chief factor of $G \in \mathfrak{Z}(\mathfrak{X})$, we have $G/C_G(N) \in f(t)$. Of course, $N \leq C_G(N)$, and so $|G/C_G(N)| < |G|$. As

$M / C_M (N) = M / M \cap C_G (N) \cong M\, C_G (N) / C_G (N) \trianglelefteq G / C_G (N) \in f (t)$,

it follows from the minimal choice of G with $(*)$ that

$M / C_M (N) \in f (t)$, as desired proving

(3) <u>$M \in \mathfrak{F}(\mathfrak{R})$</u>.

As observed above, $M / N \in f (p)$. If $t = p$ we should then have $M \in \mathfrak{S}_p\, f (p) = f (p)$. The primes p and t must therefore be different, and it follows from I.1.11 that there is a faithful, irreducible G-module, V say, over $K = GF (p)$. We have $[V] \cdot G \in \mathfrak{F}(\mathfrak{R})$ by II.3.22, and s

(4) <u>$r (V) \in \mathfrak{R}_p$</u>

when V is considered as a chief factor of V G. Were $V \mid_M$ irreducible then by (3) and (4) every chief factor of V M would be $\mathfrak{R}$-admissible. Since V is faithful for M this would lead us to conclude that $M \in f (p)$, against the choice of M. As a consequence, we apply Clifford's Theorem to give a decomposition

$$V \mid_M = V_1 \oplus \cdots \oplus V_r$$

of $V \mid_M$ into irreducible K M-modules with $r > 1$. Let χ be an absolutely irreducible character of the K G-module V, and for $i \in \{1, \ldots, r\}$, let φ_i be an absolutely irreducible character of V_i. We put $d = |K (\chi) : K|$, $e = |K (\varphi_i) : K|$ and $q = |G : M|$, a prime, noticing that if $V \mid_M$ is homogeneous then $K (\varphi_i) \subseteq K (\chi)$ by I.3.6 (d) and I.3.10(b). Set $C_i = C_M (V_i)$ and $r (V_i) = \dim_K (V_i) = a$. If $a \in \mathfrak{R}_p$ then every chief factor of $[V_i] \cdot M / C_i$ for $1 \leq i \leq r$ is $\mathfrak{R}$-admissible, forcing $M / C_i \in f (p)$ for $1 \leq i \leq r$. Since $\bigcap_{i=1}^{r} C_i = 1$ we then have $M \in R_0\, f (p) = f (p)$, a contradiction.

Hence

(5) <u>$a \notin \mathfrak{R}_p$</u>,

and the reverse argument shows that (5) is equivalent to

(5a) <u>$M / C_i \notin f (p)$ for $1 \leq i \leq r$.</u>

We aim to contradict the last two statements ((5) and (5a)); the notation introduced in the preceding paragraph is regarded as fixed for the rest of this proof.

Inspection of Clifford's Theorem if $V\big|_M$ is not homogeneous, and of I.3.12 (a) otherwise, yields

(6) $\underline{r = q \text{ or } r = \deg_q p^e}$.

We proceed by proving a series of statements, commencing with

(7) $\underline{r \neq \deg_q p^e}$.

Suppose to the contrary that $r = \deg_q p^e$. This can only happen if $V\big|_M$ is homogeneous, and by I.3.11 (a) we therefore have $q \neq p$. We also notice that $C_i = 1$ for $1 \leq i \leq r$. Let $Q \cong Z_q$. Then $Q \in \mathcal{Q}(G) \subseteq \mathfrak{f}(p) \subseteq \mathfrak{F}(\mathfrak{X})$. Let T be a non-trivial irreducible submodule of the $K(M \times Q)$-module $V \otimes_K W$. By I.1.7 the dimension of W is $\deg_q p$ and if η is an absolutely irreducible character of W then $|K(\eta) : K| = \deg_q p$. From I.4.12 we get

$$\begin{aligned}\dim_K T &= \big[\,|K(\varphi_1) : K|\,,\ |K(\eta) : K|\,\big].r_a(V_1).1\\ &= [e\,,\ \deg_q p\,].r_a(V_1)\\ &= \deg_q p^e\,.\ |K(\varphi_1) : K|\,.\ r_a(V_1)\\ &= \deg_q p^e.r(V_1) = r(V) \in \mathfrak{X}_p.\end{aligned}$$

Since $M \times Q \in \mathfrak{F}(\mathfrak{X})$ it therefore follows that every chief factor of $[T]\cdot(M \times Q) \,/\, \mathrm{Ker}(M \times Q \text{ on } T)$ is $\mathfrak{X}$-admissible and so, with $J = M \times Q \,/\, \mathrm{Ker}(M \times Q \text{ on } T)$, we have $J \in \mathfrak{f}(p)$. By I.4.13, the group J is a central product of M with Q. It is now evident that $M \in \mathcal{Q}(J) \subseteq \mathfrak{f}(p)$, contradicting our choice of M and proving (7).

(8) $\underline{\text{Either } p = q \text{ or } p \neq q = 2}$.

Suppose to the contrary that $p \neq q$ and $q \neq 2$. Let E be an extraspecial group of order q^3 and let U be a faithful irreducible E-module over K. By I.4.7 the dimension of U is $q \deg_q p$ and if

Ψ is an absolutely irreducible character of U then $|K(\Psi) : K| = \deg_q p$ by I.4.8.

<u>Case 8a : $V|_M$ is homogeneous</u>. By I.3.12 (c) we have $\deg_q p \;\Big|\; |K(\varphi_1) : K|$. Let T be an irreducible submodule of the $K(M \times E)$-module $V_1 \otimes_K U$, noticing that $C_i = 1$ for $1 \leqslant i \leqslant r$ in this case. By I.4.12 we have

$$\begin{aligned}\dim_K T &= \big[|K(\varphi_1) : K|, \deg_q p\big] \cdot r_a(V_1) \cdot q \\ &= |K(\varphi_1) : K| \cdot r_a(V_1) \cdot q \\ &= r(V_1) \cdot q = r(V) \in \mathfrak{R}_p.\end{aligned}$$

Hence $[T] \cdot (M \times E) / \mathrm{Ker}\,(M \times E \text{ on } T) \in \mathfrak{J}(\mathfrak{R})$ and $J = (M \times E) / \mathrm{Ker}\,(M \times E \text{ on } T)$ belongs to $\mathfrak{F}(p)$. By I.4.13 the group J is a central product of M and E. If $(|Z(M)|, q) = 1$ then $J = M \times E$, implying that $M \in Q(J) \subseteq \mathfrak{F}(p)$, against (5a). So $q \;\Big|\; |Z(M)|$. By assumption $q \neq 2$, and so, since $Z(M)$ and $Z(Q)$ are cyclic, there exist at least two distinct central subgroups A_1 and A_2 of $M \times E$ satisfying

(i) $A_j \cong Z_q$,

(ii) $M \times E / A_j \cong J$, and

(iii) $A_1 \cap Z(M) = 1 = A_2 \cap Z(E)$

for $j \in \{1, 2\}$. In particular, $M / A_j \in \mathfrak{F}(p)$ for $j = 1,2$, and since $A_1 \cap A_2 = 1$ we conclude that $M \in QR_0\, \mathfrak{F}(p) = \mathfrak{F}(p)$, again contradicting the choice of M.

<u>Case 8b : $V|_M$ is not homogeneous</u>. Since $Z_q \in Q(G) \subseteq \mathfrak{F}(p)$ we have $Q \times G \in \mathfrak{F}(p)$, where $Q \cong Z_q$. Let W be a non-trivial irreducible $M \times Q$-module over K and let η be an absolutely irreducible character of W. Then by I.1.7, $\dim_K W = |K(\eta) : K| = \deg_q p$. Let S be an irreducible submodule of the $K(G \times Q)$-module $V \otimes_K W$. By I.4.12 then,

$$\dim_K S = \big[|K(\chi) : K|, |K(\eta) : K|\big] \cdot r_a(V) \cdot 1$$

$$= \left[d \, , \, \deg_q p \right] \cdot r_a(V) \cdot 1$$

$$= \deg_q p^d \cdot |K(\chi) : K| \cdot r_a(V)$$

$$= \deg_q p^d \cdot r(V)$$

Since $G \times Q \in f(p)$ we have $[S] \cdot (G \times Q) / \operatorname{Ker}(G \times Q \text{ on } S) \in \mathfrak{J}(\mathfrak{R})$ and so

$$r(S) = \deg_q p^d . r(V) \in \mathfrak{R}_p .$$

Now let T be an irreducible submodule of the $K(M/C_1 \times E)$-module $V_1 \otimes_K U$. Again appealing to I.4.12, and noticing that $K(\varphi_1) = K(\chi)$ by I.3.3, we get

$$\dim_K T = \left[|K(\varphi_1) : K| \, , \, |K(\psi) : K| \right] . \, r_a(V_1) \cdot q$$

$$= \left[|K(\chi) : K| \, , \, \deg_q p \right] r_a(V_1) \cdot q$$

$$= \deg_q p^d \cdot |K(\chi) : K| \cdot r_a(V_1) \cdot q$$

$$= \deg_q p^d \cdot |K(\varphi_1) : K| \cdot r_a(V_1) \cdot q$$

$$= \deg_q p^d \cdot r(V_1) \cdot q$$

$$= \deg_q p^d \cdot r(V) \in \mathfrak{R}_p .$$

As in Case 8a, using I.4.13, we now deduce that a central product of M / C_1 with E lies in $f(p)$, implying that $M / C_1 \in f(p)$ and contradicting (5a). Thus (8) is proven.

(9) <u>G is not primitive.</u>

Suppose to the contrary that G is primitive, and let H be a complement of N in G. We set $M_0 = M \cap H$ and recall that t is the prime divisor of $|N|$. Then $H \in f(t)$ and the minimality of G with (✱) forces $M_0 \in f(t)$. By II.1.18 there is an integer n such that $[N \times \underset{n}{\cdots} \times N] \cdot H$ is faithfully and irreducibly represented on a module, X say, over K of dimension $m' \, |H|$, where $m' \mid n = \deg_t p$. Let $N^* = N \times \underset{n}{\cdots} \times N$, the socle of $[N \times \underset{n}{\cdots} N] \cdot H$. Considering $X|_{N^*}$ and applying Clifford's

Theorem, we see that $m \mid \dim_K X$. Let $q_1, \ldots, q_k$ be the set of distinct primes in $\mathbb{P} \setminus \{t, q\}$ such that

$\dim_K X = m\, q^b\, q_1^{b_1} \cdots q_k^{b_k}\, t^c$ with $b_1, \ldots, b_k \in \mathbb{N}$ and $b, c \in \mathbb{N} \cup \{0\}$. Let $q^l \,\|\, |H|$. Then $b = 0$ only if $q = t$ or $q^l \mid m / m'$ (and these events are mutually exclusive !). Since $N^* H \in R_o(G) \subseteq f(p)$ we have from II.3.22 that $\dim_K X \in \mathfrak{R}_p$. Notice also that

$$N^* M_o \in R_o(N M_o) = R_o(M) \subseteq \mathfrak{Z}(\mathfrak{R}).$$

We consider $X \mid_{N^* M_o}$. If $X \mid_{N^* M_o}$ is an irreducible $K N^* M_o$-module then it must even be faithful for $N^* M_o$. In particular we have $[X] \cdot N^* M_o \in \mathfrak{Z}(\mathfrak{R})$ and hence $M \in Q(N^* M_o) \subseteq f(p)$. So $X \mid_{N^* M_o}$ must reduce. Choose an irreducible $K N^* M_o$-submodule, X^* say, of X and let φ^* be an irreducible character of X^*. Set $K(\varphi^*) = GF(p^f)$. If $X \mid_{N^* M_o}$ were the sum of $\deg_q p^f$ copies of X then we could use the argument employed in the proof of (7) to deduce that $N^* M_o$, and hence M, lies in $f(p)$, against the choice of M. Inspection of Clifford's Theorem and I.3.12 (a) leads us to the conclusion that $X \mid_{N^* M_o}$ is a direct sum of q irreducible $K N^* M_o$-modules, and so $\dim_K X = q . \dim_K X^*$. The faithfulness of X for $N^* H$ means that N^* may not act trivially on X^*, so by Clifford's Theorem and I.1.7 we have $m \mid \dim X^*$ and therefore $q \mid \dim_K X / m$. We can therefore discount the possibility that $q^l \mid m / m'$, and so

$$b = 0 \text{ only if } q = t.$$

Since $|H| < |G|$ we have $S_n(H) \subseteq f(t)$. Thus if S_i / T_i is a q_i-chief factor of H for $1 \leqslant i \leqslant k$ we find that

$$Z_{q_i} \in Q(S_i) \subseteq f(t).$$

Similarly,

$$Z_q \in f(t).$$

Hence by II.3.21, we see that $E(q_i, t) \in \mathfrak{J}(\mathcal{R})$ for $1 \leq i \leq k$, and $E(q, t) \in \mathfrak{J}(\mathcal{R})$ if $q \neq t$. Set $E_i = E(q_i, t)$, let U_i be a faithful, irreducible E_i-module over K for $1 \leq i \leq k$, and let Ψ_i be an absolutely irreducible character of U_i, $1 \leq i \leq k$. By I.4.10 we have

$$\dim_K U_i = n_i q_i$$

where $n_i \mid \deg_t p$, and by I.4.11,

$$|K(\Psi_i) : K| = n_i.$$

So an irreducible $K(\Psi_i)E_i$-module affording Ψ_i has dimension q_i.

Let $g \in \mathbb{N}$ be such that $t^g \| p^m - 1$ and define Z to be a cyclic group of order s, where s is defined as follows:

$$s = \begin{cases} t^{g+c} & \text{if } t \neq 2, \\ t^c & \text{if } t = 2 \text{ and } c \in \{0, 1\}, \\ t^{g+1} & \text{if } t = 2 \text{ and } c = 2, \\ t^{g+h+c-2} & \text{if } t = 2,\ 2^h \| p+1 \text{ and } c > 2. \end{cases}$$

Appealing to I.1.6 (c) and I.1.7 we see that Z has a faithful and irreducible module Y of dimension $m\,t^c$ over K. If ζ is an absolutely irreducible character of Y then $|K(\zeta) : K| = m\,t^c$.

Set

$$J_1 = Z \times \mathop{X}_{j=1}^{k} \mathop{X}_{i=1}^{b_j} E_j,$$

and let T_1 be an irreducible submodule of the KJ_1-module

$$W_1 = Y \otimes_K \bigotimes_{j=1}^{k}{}_K \bigotimes_{i=1}^{b_j}{}_K\ U_j.$$

By I.4.12,

$$\dim_K T_1 = \Big[\,|K(\zeta) : K|\,,\ \text{l.c.m.}\big\{|K(\psi_i) : K| \ :\ 1 \leq i \leq k\big\}\Big].\ \prod_{j=1}^{k}\prod_{i=1}^{b_j} q_j$$

$$= |K(\zeta) : K|.\ q_1^{b_1} \cdots q_k^{b_k}$$

$$= m\, t^c\, q_1^{b_1} \cdots q_k^{b_k}.$$

If $b = 0$ then $\dim_K T_1 = \dim_K X \in \mathfrak{X}_p$. Otherwise we have $q \neq t$. Let $\Xi = \Xi(q, t)$ and let U be a faithful, irreducible Ξ-module over K with absolute irreducible character ψ. Arguing as above, we see that an irreducible submodule T_2 of the $K(\Xi \times \underset{b}{\cdots} \Xi \times J_1)$-module

$U \otimes_K \cdots \otimes_K U \otimes_K W_1$ has dimension $m\, q^b\, q_1^{b_1} \cdots q_k^{b_k}\, t^c$. We set

$$J = \begin{cases} J_1 & \text{if } b = 0 \\ J_2 & \text{if } b \neq 0 \end{cases}$$

$$T = \begin{cases} T_1 & \text{if } b = 0 \\ T_2 & \text{if } b \neq 0 \end{cases}$$

and so we have $\dim_K T = \dim_K X \in \mathfrak{X}_p$. By construction $J \in \mathfrak{J}(\mathfrak{X})$, and so every chief factor of $[T] \cdot J$ is $\mathfrak{X}$-admissible. It is easy to apply I.4.13 and deduce that T is faithful for J. Hence

$$[T]\cdot J \in \mathfrak{J}(\mathfrak{X}) \text{ and } J \in f(p).$$

Now let $\bar{Z}$ be a cyclic group of order $|Z|\,/\,t$ if $Z \neq 1$, and $\bar{Z} = Z$ otherwise. Define $\bar{J} \in Q(J) \subseteq f(p)$ by

$$\bar{J} = \begin{cases} \bar{Z} \times \mathop{\times}\limits_{j=1}^{k} \mathop{\times}\limits_{i=1}^{b_j} \Xi_j & \text{if } b = 0 \\[2ex] \Xi \times \underset{b-1}{\cdots} \times \Xi \times J_1 & \text{if } b \neq 0. \end{cases}$$

A now familiar argument using I.4.12 and I.4.13 proves the existence of a faithful irreducible $\overline{J}$-module $\overline{T}$ over K of dimension

$$\dim_K \overline{T} = \begin{cases} m\, q_1^{b_1} \cdots q_k^{b_k}\, q^{c-1} & \text{if } b = 0 \\ m\, q^{b-1}\, q_1^{b_1} \cdots q_k^{b_k}\, t^c & \text{if } b \neq 0. \end{cases}$$

Therefore

$$\dim_K \overline{T} = \dim_K T \,/\, q = \dim_K X \,/\, q = \dim_K X^{*}.$$

Since $\overline{J} \in f(p)$ it follows that that $[\overline{T}] \cdot \overline{J} \in \mathfrak{J}(\mathcal{R})$ and hence $\dim_K X^{*} = \dim_K \overline{T} \in \mathcal{R}_p$. This means that $[X^{*}] \cdot N^{*} M_o \in \mathfrak{J}(\mathcal{R})$ since all its chief factors are $\mathcal{R}$-admissible, and thus

$$N^{*} M_o \,/\, \operatorname{Ker}(N^{*} M_o \text{ on } X^{*}) \in f(p).$$

Now, X^{*} was arbitrarily chosen as an $N^{*} M_o$-submodule of X and because X is faithful for $N^{*} M_o$ we deduce that

$$N^{*} M_o \in R_o\, f(p) = f(p),$$

This yields the contradiction

$$M = N M_o \in Q(N^{*} M_o) \subseteq f(p),$$

proving (9).

(10) $Z_t \in f(p)$; if $t \neq s \in \mathbb{P}$ and $s \mid |G|$ then $E(s, t) \in \mathfrak{J}(\mathcal{R})$ if $p \neq s \in \mathbb{P}$ and $s \mid |G|$, then $\deg_s r^d \cdot r(V) \in \mathcal{R}_p$.

If G is a t-group then $Z_t \in Q(G) \subseteq f(p)$, so let s be a prime such that $t \neq s \mid |G|$. Since N is the unique minimal normal subgroup of G we have $O_t(G) = F(G)$. Thus by II.1.17

there is a complemented t-chief A / B of G on which a Sylow s-subgroup of G acts non-trivially. By II.1.12 we have

$$J = [A / B] \cdot A_G (A / B) \in Q(G) \subseteq f(p)$$

and $s \mid |A_G (A / B)|$ by choice of A / B. Since G is not primitive, we must have $|J| < |G|$ and so by the minimal choice of G with (✱) we see that $S_n (J) \subseteq f(p)$. In particular,

$$Z_t \in Q(F(J)) \subseteq f(p).$$

Similarly, $|A_G (A / B)| < |G|$ and so $S_n (A_G (A / B)) \subseteq f(t)$. Since $s \mid |A_G (A / B)|$, there is an s-chief factor X / Y of $A_G (A / B)$ and then we have

$$Z_s \in Q(X) \subseteq f(t),$$

and so

$$E(s, t) \in \mathfrak{J}(\mathfrak{R})$$

by II.3.22.

Now suppose that s was chosen so that $p \nmid s \mid |G|$ and set $F = \mathrm{Soc}(J)$ with A / B and J as above. Then FX / FY is an s-chief factor of J and we have

$$Z_s \in Q(FX) \subseteq f(p)$$

since $FX \trianglelefteq J$. Then $G \times Z_s \in f(p)$.

We now have $G \times S \in f(p)$, where $S \cong Z_s$. Let W be a non-trivial irreducible S-module over K, and let η be an absolutely irreducible character of W. Let T be an irreducible submodule of the $K(G \times S)$-module $V \otimes_K W$. By I.1.7 we deduce that

$$\dim_K W = |K(\eta) : K| = \deg_s p.$$

Using I.4.12, therefore, we have

$$\begin{aligned}\dim_K T &= \left[|K(\chi) : K|, \deg_s p\right] \cdot r_a(V)\\ &= \deg_s p^d \cdot |K(\chi) : K| \cdot r_a(V)\\ &= \deg_s p^d \cdot r(V).\end{aligned}$$

Since $G \times S \in f(p)$ we have $[T] \cdot (G \times S)^{*} / \mathrm{Ker}\,(G \times S \text{ on } T) \in \mathfrak{J}(\mathfrak{R})$, and so

$$\dim_K T = \deg_s p^d . r(V) \in \mathfrak{R}_p .$$

(11) <u>G is not a q-group.</u>

Suppose that G is a q-group. By I.4.5 then, $r(V) = q^b . \deg p$ for some b. Let g be the positive integer such that $q^g \mid p^c - 1$, where we set $c = \deg_q p$. Define Z to be a cyclic group of order q^s, where s is defined as follows:

$$s = \begin{cases} g + b & \text{if } q \neq 2 \\ b & \text{if } q = 2 \text{ and } b \in \{0, 1\} \\ g + 1 & \text{if } q = 2 \text{ and } b = 2 \\ g + h + b - 2 & \text{if } q = 2,\ 2^h \mid p + 1 \\ & \text{and } b > 2. \end{cases}$$

It is easily checked, using I.1.7 and I.1.6 (c) that Z is faithfully and irreducible represented on a module W of dimension $c.q^b$ over K. Since $c.q^b = r(V) \in \mathfrak{R}_p$, every chief factor of $[W] \cdot Z$ is $\mathfrak{R}$-admissible, and we have

$$Z \in f(p).$$

As we require that $q \mid r(V)$, we can ignore the possibility that $b = 0$. Define $\bar{Z} \in Q(Z)$ to be the cyclic group of order $|Z| / q$. Then $\bar{Z} \in f(p)$ and is faithfully and irreducibly represented on a module $\bar{W}$ of dimension $c.q^{b-1}$ over K. Since we must have $[\bar{W}] \cdot \bar{Z} \in \mathfrak{J}(\mathfrak{R})$ we conclude that

$$c.q^{b-1} = r(V) / q = r(V_1) \in \mathfrak{R}_p ,$$

against (5).

(12) <u>if $p \neq s \in \mathbb{P}$ and $s \mid |G|$, then $\deg_s p^e = \deg_s p^d$.</u>

Suppose to the contrary. Since $d = q.e$ by I.3.12 (b), we deduce that $\deg_s p^e = q.\deg_s p^d$. Let $S \cong Z_s$ and let W be a non-trivial irreducible K S-module. By I.1.7, $\dim_K W = \deg_s p$. Let T be an irreducible submodule of the K (M / C_1 x S)-module $V_1 \otimes_K W$. By I.4.12 we have

$$\dim_K T = [|K(\varphi_1) : K|, \deg_s p] \cdot r_a(V_1) \cdot 1$$

$$= \deg_s p^e \; |K(\varphi_1) : K| \cdot r_a(V_1)$$

$$= q \deg_s p^d \, r(V_1)$$

$$= \deg_s p^d \cdot r(V) \in \mathfrak{N}_p \text{ (by (10))}.$$

Thus every chief factor of $[T] \cdot (M / C_1 \times S) / \mathrm{Ker}(M / C_1 \times S \text{ on } T)$ is $\mathfrak{A}$-admissible, and we see that $L / \mathrm{Ker}(L \text{ on } T) \in f(p)$, where we set $L = M / C_1 \times S$. By I.4.13 the group $L / \mathrm{Ker}(L \text{ on } T)$ is a central product of $M / C_1 \times S$ and so we have that

$$M / C_1 \in Q(L / \mathrm{Ker}(L \text{ on } T)) \subseteq f(p),$$

against (5a).

(13) <u>$p \neq q$.</u>

Suppose that $p = q$. Then $V|_M$ cannot be homogeneous by I.3.11 (a). So by I.3.3 we have $K(\chi) = K(\varphi_1)$. Since $p \neq t$ it follows from (10) that $E(p, t) \in \mathfrak{I}(\mathfrak{R})$. Let $E = E(p, t)$ and let U be a faithful, irreducible E-module over K with absolutely irreducible character Ψ. Let $S \cong Z_t$ and let W be a non-trivial irreducible S-module. By I.4.10 and I.4.11 we have

$$\dim_K U = n.p, \text{ where } n \mid \deg_t p$$

and

$$|K(\Psi) : K| = n.$$

Since $\dim_K W = \deg_t p$ it follows from I.4.12 that if T is an irreducible submodule of the K (M / C_1 x E x S)-module

$V_1 \otimes_K U \otimes_K W$ then

$$\dim_K T = \left[|K(\varphi_1) : K| , [n, \deg_t p] \right] . r_a (V_1) . p$$

$$= \left[|K(\varphi_1) : K| , \deg_t p \right] . r_a (V_1) . p$$

$$= \deg_t p^e . |K(\varphi_1) : K| . r_a (V_1) . p$$

$$= \deg_t p^d . r (V_1) . p \quad \text{(using (12))}$$

$$= \deg_t p^d . r (V) \in \mathfrak{R}_p \text{ by (10).}$$

Setting $L = M / C_1 \times E \times S$ the usual argument using I.4.13 shows that $M / C_1 \in Q (L / \mathrm{Ker} (L \text{ on } T)) \subseteq f (p)$, against (5a).

(14) G is a $\{2, p\}$-group.

By (8) and (13) we have $p \neq q = 2$. Suppose that $s \in \mathbb{P} \setminus \{2, p\}$ and $s \mid |G|$. Since $\mathfrak{R}_2 \neq \emptyset \neq \mathfrak{R}_s$ it follows from III.1.1 that $E (2, s) \in \mathfrak{Z} (\mathfrak{R})$. Let $E = E (2, s)$, let U be a faithful irreducible E-module over K and let ψ be an absolutely irreducible character of U. Let W be a non-trivial irreducible $S \cong Z_s$-module. Then

$$\dim_K U = 2 n \text{ where } n \mid \deg_s p$$

and

$$|K(\psi) : K| = n.$$

If T is an irreducible submodule of the $K (M / C_1 \times E \times S)$-module $V_1 \otimes_K U \otimes_K W$, then arguing as in (13) we see that $\dim_K T = \deg_s p^d r (V) \in \mathfrak{R}_p$, and hence that $M / C_1 \in f (p)$, again contradicting (5a).

(15) $E (p, 2) \in f (p)$.

Let $Q = O^{2', 2} (G)$. Since $C_p (G) = 1$ and G is not a 2-group we must have $1 \neq Q$. Let Q / B be a chief factor of G. Evidently

$2 \mid |Q/B|$, and, by construction, we must have $O_p(A_G(Q/B)) \neq 1$. By II.3.8 (b) we have $J = [Q/B] \cdot A_G(Q/B) \in f(p)$. Since G is not primitive we deduce that $|J| < |G|$, and so $S_n(J) \subseteq f(p)$. If P is a minimal normal p-subgroup of $A_G(Q/B)$ then $F(J)P \in f(p)$ and it follows easily that

$$E(p, 2) \in Q(F(J)P) \subseteq f(p).$$

(16) $O^{2'}(G) = G$.

Suppose to the contrary and let $M^* \trianglelefteq \cdot G$ with $|G : M^*| = p$. Then (13) we infer that $M^* \in f(p)$. Let $L = M \cap M^*$. Since $L \trianglelefteq M^*$ and $S_n(M^*) \subseteq f(p)$, we have $L \in f(p)$. Thus if V^* is an irreducible submodule of $V_1 |_L$, then $r(V^*) \in \mathcal{R}_p$. In particular, $V_1 |_L$ must be reducible, and, by I.3.11 (a), cannot be homogeneous. By Clifford's Theorem, therefore, $p.r(V^*) = r(V_1)$. Let U be a faithful irreducible $E = E(p, 2)$-module over K. It is easily checked from I.4.10 and I.4.11 that U is absolutely irreducible and has dimension p. By I.4.12 then, $V^* \otimes_K U$ is an irreducible $K(L / \mathrm{Ker}(L \text{ on } V^*) \times E)$-module of dimension $r(V^*).p = r(V_1)$. Since $L / \mathrm{Ker}(L \text{ on } V^*) \times E \in f(p)$, the implication now is that $r(V_1) \in \mathcal{R}_p$, contrary to (5).

(17) The final contradiction.

Again let U be a faithful irreducible $E = E(p, 2)$-module over K. Then $V \otimes_K U$ is an irreducible $G \times E$-module over K, and so, as $G \times E \in f(p)$, we infer that $\dim_K(V \otimes_K U) = p.\dim_K V \in \mathcal{R}_p$.

Let $Q = O^{2', 2}(G)$. By (11) and (16) and because $t \neq p$ we have

$$1 < Q < O^2(G) < G.$$

Let Q / B be a chief factor of G. Clearly, $2 \mid |Q/B|$. Let $\mathfrak{N}(2)$ be the class of 2-nilpotent groups. One sees easily that $\mathfrak{N}(2)$ is a formation and that $Q = G^{\mathfrak{N}(2)}$. Since Q / B is an abelian normal subgroup of G / B and $(Q / B)^{\mathfrak{N}(2)} = Q / B$ by II.5.3, it follows from [4], Theorem 5.13 that Q / B is complemented in G / B, and hence that Q / B is a complemented chief factor of G. Hence $[Q / B] \cdot A_G (Q / B) \in Q(G)$. If $p \nmid |A_G (Q / B)|$ then we have $A_G (Q / B) = 1$ and if H is a maximal subgroup of G complementing Q / B, then $H \triangleleft \cdot G$. We further have $Q H = G$ and $Q \cap H = B$. But now, if $P \in Syl_p (G)$ then $B P / B \trianglelefteq G / B$, for $B P \leq H$, $G / B = Q / B \times H / B$, and, clearly, $B P / B = O^{2} (H / B)$. Now, $G / B P$ is a 2-group. But this implies that $Q = O^{2', 2}(G) \leq B$, a contradiction. Therefore $p \mid |A_G (Q / B)|$. Since $A_G (Q / B) \in f(2)$ and $O^{2'} (A_G (Q / B)) = A_G (Q / B)$, we find that there is a non-trivial p-chief factor $O^{2} (A) / D$ of $A = A_G (Q / B)$, and using the Schur-Zassenhaus theorem we deduce that this chief factor is complemented in A. Hence

$$L = [O^{2} (A) / D] \cdot A / C_A (O^{2} (A) / D) \in Q (A) \subseteq f(2).$$

Since $A = O^{2'} (A)$, the quotient $A / C_A (O^{2} (A) / D)$ is a non-trival 2-group. We can now use the fact that $S_n(L) \subseteq f(2)$ to deduce that $E (2 , p) \in f(2)$. Let Y be a faithful, irreducible $E (2 , p)$-module over GF (2) and let $F = [Y] \cdot E (2 , p) \in \mathfrak{F}(\mathfrak{R})$. By II.1.1[illegible] there is an integer n such that $F^{*} = [Y \times \cdot_n \cdot \times Y] \cdot E (2 , p)$ is faithfully and irreducibly represented on a module [illegible] of dimension $2p$ over [illegible], and this module must be absolutely irreducible. Hence by I.4.12, $K (M / C_1 \times F^{*})$ is irreducibly represented on the module $V_1 \bigcirc_K$ [illegible] of dimension $r (V_1) \cdot 2 p = p \cdot r (V) \in \mathfrak{R}_p$. Now, $F^{*} \in R_0 (F) \subseteq \mathfrak{F}(\mathfrak{R})$, so every chief factor of $[V_1 \bigcirc_K$ [illegible]$] \cdot M / C_1 \times F^{*}$ is $\mathfrak{R}$-admissible (observe that $V_1 \bigcirc_K$ [illegible] is faithful for $M / C_1 \times F^{*}$ by I.4.13) and so firstly

$$[V_1 \bigcirc_K \ldots] \cdot M / C_1 \times F^{*} \in \mathfrak{F}(\mathfrak{R})$$

and hence, secondly,

$$M / C_1 \times F^* \in f(p).$$

But now this contradicts (5a) and the proof is complete.

q.e.d.

III.1.3 Corollary If t is a prime, $G \in f(p)$ and $t \mid |G|$, then $Z_t \in f(p)$.

Proof There is a t-chief factor X / Y of G. Then

$$Z_t \in Q(X) \subseteq f(p);$$

for since $X \trianglelefteq G$, we have $X \in f(p)$ by III.1.2.

q.e.d.

With Theorem III.1.2 behind us we can make rapid progress. We need some definitions:

III.1.4 Definitions (a) If m is a positive integer such that $\mathfrak{S}_p \neq \emptyset$ whenever $p \in \mathbb{P}$ and $p \mid m$, then we say that m is $\mathfrak{S}$-potent.

(b) Let m and f be positive integers and p be a prime. If m_a and m_b are positive integers such that m_b is square-free, $p \nmid m_b$ and $m = m_a \deg_{m_b} p^f$, then we call the (ordered) pair m_a, m_b a p^f-factorisation of m, and denote this pair by $\{m_a, m_b\}_{p^f}$. We name m_a a p^f-absolute part of m and m_b a p^f-base part of m.

III.1.5 Proposition. A positive integer m belongs to $\mathfrak{R}p$ for some prime p if and only if it has the following properties:

(a) there exist $\mathfrak{R}$-potent integers m_a and m_b such that $\{m_a, m_b\}_p$ is a p-factorisation of m.

(b) If $q \in \mathbb{P}$ and $q \mid m_a$, then either

(i) $q \mid m_b$, or

(ii) there is a prime r such that $r \mid m_b$ and

$\Xi(q, r) \in \mathfrak{J}(\mathfrak{R})$ - equivalently,

$\deg_q r \in \mathfrak{R}_r$. In fact, $\Xi(q, r) \in f(p)$

for all such r.

(c) $\deg_{m_b} p \in \mathfrak{R}_p$ and, equivalently, $Z_{m_b} \in f(p)$.

(d) If $r \mid (m_a, m_b)$ then f (p) contains every extraspecial r-group. In particular,

$$r^h \deg_r p \in \mathfrak{R}_p \text{ for } h \geqslant 0.$$

Proof Suppose first that $m \in \mathfrak{R}_p$ for some prime p. Because of the assumed minimality of $\mathfrak{R}$, there is a group $G \in f(p)$ having a faithful, irreducible module, V say, over $K = GF(p)$ of dimension m. By I.4.4 there is a p-decomposition $\{m_a, m_b\}_p$ of m such that:

(1) $\exp(\mathrm{Soc}(G)) \mid m_b$;

(2) $p \nmid m_b$;

(3) m_b is square-free;

(4) $m_a m_b \mid |G|$.

Since (4) holds, m_a and m_b are certainly $\mathfrak{R}$-potent (as $G \in f(p) \subseteq \mathfrak{J}(\mathfrak{R})$), so (a) is already proven.

Let

$$m_a = q_1^{c_1} \dots q_k^{c_k} r_1^{d_1} \dots r_l^{d_l}$$

be the prime decomposition of m_a, where for $1 \leqslant i \leqslant k$ we have $q_i \mid m_a$ but $q_i \nmid m_b$, whilst $r_j \mid (m_a, m_b)$ for $1 \leqslant j \leqslant l$. Fix $i \in \{1, \dots, k\}$. Since (1) holds, $q_i \nmid |\mathrm{Soc}(G)|$. Hence by II.1.17, if $S \in \mathrm{Syl}_{q_i}(G)$, then there is a complemented t_i-chief factor X / Y of G, where t_i is a prime dividing $|F(G)|$, on which S acts non-trivially. We have:

$$[X / Y] \cdot A_G(X / Y) \in Q(G) \subseteq \mathfrak{J}(\mathfrak{R})$$

and so $A_G\,(X\,/\,Y) \in f\,(t)$. Since $q_i \,\Big|\, |\dot{A}_G\,(X\,/\,Y)|$, it follows from III.1.3 that $Z_{q_i} \in f\,(t)$, whence

$$E\,(q_i\,,\,t_i) \in \mathfrak{F}(\mathfrak{R}).$$

Because $t_i \,\Big|\, |\mathrm{Soc}\,(G)|$ we certainly have $t_i \,\Big|\, m_b$ by (1) above. The first part of (b) is now proven.

To each prime q_i we associate a prime t_i, $1 \leqslant i \leqslant k$, with $t_i \,\Big|\, m_b$ and $E\,(q_i\,,\,t_i) \in \mathfrak{F}(\mathfrak{R})$. Set $E_i = E\,(q_i\,,\,t_i)$ and let U_i be a faithful, irreducible E_i-module over K for $1 \leqslant i \leqslant k$. Let Ψ_i be an absolutely irreducible character of U_i. By I.4.10 and I.4.11 there is an integer n_i for $1 \leqslant i \leqslant k$ with $n_i \,\Big|\, \deg_{t_i} p$ and:

$$\dim_K U_i = n_i . q_i, \text{ and}$$

$$|K(\Psi_i) : K| = n_i.$$

For $1 \leqslant j \leqslant l$ let P_j be an extraspecial group of order $r_j^{2d_j+1}$.

From the definition of the r_j's and (2) we have $r_j \neq p$, and because P_j has a unique minimal normal subgroup we can therefore find a faithful irreducible P_j-module, W_j, over K for $1 \leqslant j \leqslant l$. If η_j is an absolutely irreducible character of W_j then it follows from I.4.7 and I.4.8 that

$$\dim_K W_j = r_j^{d_j} \deg_{r_j} p, \text{ and}$$

$$|K(\eta_j) : K| = \deg_{r_j} p.$$

Because r_j is $\mathfrak{R}$-potent, it follows from III.1.1 that $1 \in \mathfrak{R}_{r_j}$ and so $P_j \in \mathfrak{F}(\mathfrak{R})$ for $1 \leqslant j \leqslant l$. We set $m'_b = m_b \,/\, r_1 \ldots r_l$. Again since m'_b is $\mathfrak{R}$-potent we have $Z_{m'_b} \in \mathfrak{F}(\mathfrak{R})$. Hence the group H defined below is a direct product of groups in $\mathfrak{F}(\mathfrak{R})$, and so is itself an element of $\mathfrak{F}(\mathfrak{R})$:

$$H = Z_{m_b'} \times \mathop{\mathsf{X}}_{j=1}^{l} P_j \times \mathop{\mathsf{X}}_{i=1}^{k} \mathop{\mathsf{X}}_{j=1}^{c_i} E(q_i, t_i).$$

Let Y be a non-trivial irreducible $Z_{m_b'}$-module over K and let T be an irreducible submodule of the K H-module defined as follows:

$$Y \otimes_K \bigotimes_{j=1}^{l} {}_K W_j \otimes_K \bigotimes_{i=1}^{k} {}_K \bigotimes_{j=1}^{c_i} {}_K U_i$$

Iterative application of I.4.12 gives

$$\begin{aligned} \dim_K T &= \left[\deg_{m_b'} p, \{\deg_{r_j} p : 1 \leq j \leq l\}, \{n_i : 1 \leq i \leq k\}\right] \prod_{j=1}^{l} r_j^{d_j} \prod_{i=1}^{k} q_i^{c_i} \\ &= \deg_{m_b} p \cdot m_a \\ &= m. \end{aligned}$$

By I.4.13 the module T is faithful for H. Each chief factor of $[T]\cdot H$ is $\mathfrak{R}$-admissible, so $TH \in \mathfrak{F}(\mathfrak{R})$ and $H \in f(p)$. Hence

$$(Z_{m_b'}) \cup (P_j : 1 \leq j \leq l) \cup (E(q_i, t_i) : 1 \leq i \leq k) \subseteq Q(H) \subseteq f(p).$$

The proof of (b) is now complete. Since $P_j \in f(p)$ for $1 \leq j \leq l$ and $Z_{r_j} \in Q(P_j) \subseteq f(p)$, we have

$$Z_{m_b'} \times Z_{r_1} \times \ldots \times Z_{r_l} \cong Z_{m_b} \in f(p),$$

proving (c).

To prove (d), we observe that by [26], III.13.7, III.13.8, there is a normal subgroup $\overline{P}_j$ of P_j which is extraspecial of order r_j^3. Then $\overline{P}_j \in S_n(P_j) \in f(p)$ by III.1.2. Again by [26], III.13.7, III.13.8, with suitable choice of P_j we get every extraspecial group of order r_j^3 in $f(p)$. The same citation then ensures that, since every extraspecial r_j-group is a central product of certain extraspecial r_j-groups of order r_j^3, every extraspecial r_j-group belongs to $f(p)$.

The reverse implication is easy, since the group H defined above belongs to $f(p)$ by (b), (c) and (d). Then, as above, there is a faithful, irreducible H-module T of dimension m. Since then $[T]\cdot H \in \mathfrak{F}(\mathcal{R})$ we have $m \in \mathcal{R}_p$.
q.e.d.

The existence of a p-factorisation for each element of $\mathcal{R}_p$ with certain of the properties listed in III.1.5 is crucial. We wish to formalise this notion for future reference.

<u>III.1.6 Definitions</u> Let $p \in \mathbb{P}$ and $m \in \mathbb{N}$. We shall say that m is <u>$\mathcal{S}_p$-integrated</u> if there is a p-factorisation $\{m_a, m_b\}_p$ of m in which m_a and m_b are $\mathcal{S}$-potent such that if $q \in \mathbb{P}$ and $q \mid m_a$ but $q \nmid m_b$ then there exists $r \in \mathbb{P}$ with $r \mid m_b$ such that $\deg_q r \in \mathcal{S}_r$. Such p-factorisation will be said to be <u>$\mathcal{S}_p$-integral</u> and referred to as an <u>$\mathcal{S}_p$-factorisation</u> of m. Evidently, if $\{m_a, m_b\}_p$ is an $\mathcal{S}_p$-factorisation of m and $m \neq 1$, then $m_b \neq 1$. If every element of $\mathcal{S}_p$ is $\mathcal{S}_p$-integrated for all primes p, then we say that $\mathcal{S}$ is <u>integrated</u>.

The ranking function $\mathcal{R}$ is integrated by III.1.5 (a), (b) and (c).

<u>III.1.7 Lemma</u> Let m be an $\mathcal{S}_p$-integrated integer with $\mathcal{S}_p$-factorisation $\{m_a, m_b\}_p$. If n is an $\mathcal{S}$-potent square-free integer such that $\deg_n p \mid m$, then $\{m / \deg_{[n, m_b]} p, [n, m_b]\}_p$ is an $\mathcal{S}_p$-factorisation of m.

Proof It is obvious that $\{m / \deg_{[n,m]} p \, , [n , m_b]\}_p$ is a p-factorisation of m, and one checks easily that it is an $\mathcal{S}_p$-factorisation from the definitions.

q.e.d.

III.1.8 Lemma Let p and q be $\mathcal{R}$-potent primes such that $q \mid m \in \mathcal{R}_p$. Let $\{m_a , m_b\}_p$ be an $\mathcal{R}_p$-factorisation of m.

(a) Every extraspecial group of order q^3 is in f (p).

(b) If $q \nmid m_b$, then there is a prime r dividing m_b such that $E(q , r) \in f(p)$.

Proof (a) If $q \mid (m_a , m_b)$, then this follows from III.1.5(d). If $p = q$ then (a) holds because f is a full local definition. If $q \mid m_a$ but $q \nmid m_b$, then by III.1 5 (c) there is a prime r dividing m_b such that $E(q,r) \in f(p)$. If $q \nmid m_a$, then $q \mid \deg_{m_b} p$, and there exists a prime r dividing m_b such that $q \mid \deg_r p$. Since then $q \mid r-1$, it follows from I.4.10 that $E(q,r)$ is represented faithfully and irreducibly on a module of dimension $\deg_r p$. By III.1.5 (c) we have $Z_r \in Q(Z_{m_b}) \subseteq f(p)$, and so $\deg_r p \in \mathcal{R}_p$. Thus in any case, $E(q,r) \in f(p)$. Then $Z_q \in Q(E(q,r)) \subseteq f(p)$, and $Z_r \in Q(F(E(q,r)))$ $\subseteq Q S_n (E(q,r)) \subseteq f(p)$. Thus we now have $E(q,r) \times Z_q \times Z_r \in f(p)$. Let U be a faithful irreducible $E(q,r)$-module over K with an absolutely irreducible character Ψ. We have $\dim_K U = nq$, where $n \mid \deg_r p$. Let W_q and W_r be non-trivial irreducible Z_q- and Z_t-modules respectively. If T is an irreducible submodule of the $K(E(q , r) \times Z_q \times Z_r)$-module $U \otimes_K W_q \otimes_K W_r$ then the usual methods show that T has dimension $q \deg_{qr} p$. Hence

$$q \deg_{qr} p \in \mathcal{R}_p.$$

Now let F be an extraspecial group of order q^3. Let Y be a faithful

Proof It is obvious that $\{m / \deg_{[n,m]} p , [n , m_b]\}_p$ is a p-factorisation of m, and one checks easily that it is an $\mathcal{S}_p$-factorisation from the definitions.

q.e.d.

III.1.8 Lemma Let p and q be $\mathfrak{R}$-potent primes such that $q \mid m \in \mathfrak{R}_p$. Let $\{m_a , m_b\}_p$ be an $\mathfrak{R}_p$-factorisation of m.

(a) Every extraspecial group of order q^3 is in f (p).

(b) If $q \nmid m_b$, then there is a prime r dividing m_b such that $E(q , r) \in f(p)$.

Proof (a) If $q \mid (m_a , m_b)$, then this follows from III.1.5(d). If $p = q$ then (a) holds because f is a full local definition. If $q \mid m_a$ but $q \nmid m_b$, then by III.1 5 (c) there is a prime r dividing m_b such that $E(q,r) \in f(p)$. If $q \nmid m_a$, then $q \mid \deg_{m_b} p$, and there exists a prime r dividing m_b such that $q \mid \deg_r p$. Since then $q \mid r-1$, it follows from I.4.10 that E (q,r) is represented faithfully and irreducibly on a module of dimension $\deg_r p$. By III.1.5 (c) we have $Z_r \in Q(Z_{m_b}) \subseteq f(p)$, and so $\deg_r p \in \mathfrak{R}_p$. Thus in any case, $E(q,r) \in f(p)$. Then $Z_q \in Q (E(q,r)) \subseteq f(p)$, and $Z_r \in Q(F(E(q,r))) \subseteq Q S_n (E(q,r)) \subseteq f(p)$. Thus we now have $E(q,r) \times Z_q \times Z_r \in f(p)$. Let U be a faithful irreducible E (q,r)-module over K with an absolutely irreducible character Ψ. We have $\dim_K U = nq$, where $n \mid \deg_r p$.

Let W_q and W_r be non-trivial irreducible Z_q- and Z_t-modules respectively. If T is an irreducible submodule of the $K(E(q , r) \times Z_q \times Z_r)$-module $U \otimes_K W_q \otimes_K W_r$ then the usual methods show that T has dimension $q \deg_{qr} p$. Hence

$$q \deg_{qr} p \in \mathfrak{R}_p.$$

Now let F be an extraspecial group of order q^3. Let Y be a faithful irreducible F-module over K. If S is an irreducible submodule of the $K(F \times Z_r)$-module $Y \otimes_K W_r$ then S also has dimension $q.\deg_{qr} p$ and S is faithful for $F \times Z_r$. Hence every chief factor of $[S] \cdot (F \times Z_r)$ is $\mathfrak{R}$-admissible, yielding $F \times Z_r \in f(p)$; in particular, $F \in f(p)$.

(b) If $q \mid m_a$ then (b) is true by III.1.5 (b). Otherwise we have $q \mid \deg_{m_b} p$ and so there is a prime r dividing m_b such that $q \mid \deg_r p$. From III.1.5 (c) we obtain $Z_r \in Q(Z_{m_b}) \subseteq f(p)$, and so

$$\deg_r p \in \mathfrak{R}_p.$$

By I.4.10, the group $E(q, r)$ is faithfully and irreducibly represented on a module V of dimension $[q, \deg_r p] = \deg_r p \in \mathfrak{R}_p$. Since $q \mid \deg_r p \mid r-1$ by I.1.6 (a) and (b), we see that $E(q, r)$ is supersoluble and so, by III.1.1, every chief factor of $[V] \cdot E(q, r)$ is $\mathfrak{R}$-admissible. Hence $E(q, r) \in f(p)$.
q.e.d.

The next lemma is our first result to give an arithmetical criterion for an integer to belong to some $\mathfrak{R}_p$.

<u>III.1.9 Lemma</u> Let $p \in \mathbb{P}$ and $m \in \mathfrak{R}_p$. If $n \mid m$ and n is $\mathfrak{Q}_p$-integrated then $n \in \mathfrak{R}_p$.

<u>Proof</u> We show that n has properties (a) - (d) of III.1.5. Let $\{n_a, n_b\}_p$ be an $\mathfrak{R}_p$-factorisation of n. Then (a) and the first part of (b) of III.1.5 are immediately satisfied. Let $\{m_a, m_b\}_p$ be an $\mathfrak{R}_p$-factorisation of m. By III.1.7, since $\deg_{n_b} p \mid m$, we may assume that $n_b \mid m_b$. Let $q \mid n_a$ with $q \in \mathbb{P}$. Then $q \mid m$ and so by III.1.8 (a) there is an extraspecial group of order q^3 in $f(p)$. Using [26] III, 13.7, III, 13.8 we see that

(d) of III.1.5 holds for n. By III.1.5(c) we have $Z_{n_t} \in Q(Z_{n_b}) \subseteq f(p)$, and so III.1.5 (c) holds for n. Now suppose that $q \mid n_a$ but $q \nmid n_b$. Since n is $\mathfrak{R}_p$-integrated, there is a prime r dividing n_b such that $\deg_q r \in \mathfrak{R}_r$. Since III.1.5 (c) holds, we have $Z_r \in Q(Z_{n_b}) \subseteq f(p)$. Also, since III.1.5 (d) holds, there is an extraspecial group E of order q^3 in f(p). Thus $E \times Z_r \in f(p)$, and we deduce ,in the usual way, that $q.\deg_{rq} p \in \mathfrak{R}_p$. Since $Z_q \times Z_r \times E\,(q,r)$ is faithfully and irreducibly represented on a module of dimension $q.\deg_{rq}$ p over GF (p) and lies in $\mathfrak{J}(\mathfrak{R})$, we infer that $E\,(q,r) \in f(p)$, as desired. q.e.d.

We have seen in Lemma III.1.9 that each set $\mathfrak{R}_p$ enjoys a certain closure property for divisors. We shall show that there is a notion of an "$\mathfrak{R}$-product" under which p is closed. Unfortunately, this operation defines a set of "$\mathfrak{R}$-products", and, in general, there is no hope of defining a unique or canonical composite of elements in $\mathfrak{R}_p$ which is again an element of $\mathfrak{R}_p$. We attempt to indicate why this is so in the following examples.

III.1.10 Examples (a) In III.1.9 we showed that if $m \in \mathfrak{R}_p$ and n is an $\mathfrak{R}_p$-integrated integer with $n \mid m$, then $n \in \mathfrak{R}_p$. One is immediately tempted to ask whether $\mathfrak{R}_p$ is, in fact, closed under taking all divisors of m. This is, however, not the case. We can also show that $\mathfrak{R}_p$ is not product closed. We recall the family of saturated formations defined in II.4.2 (e), and in particular the example preceding II.4.4. This was defined by

$$\mathcal{T}_2 = \mathcal{T}_3 = \emptyset;$$
$$\mathcal{T}_5 = \{1, 3, 6\};$$
$$\mathcal{T}_p = \mathbb{N} \text{ for all } p \geqslant 7$$

(we saw that the 2 which appeared in the original definition of $\mathcal{T}_5$ in II.4.2 (e) was redundant). Although $\mathcal{T}$ is not minimal for $\mathfrak{F}(\mathcal{T})$ (for instance, the 2 in $\mathcal{T}_7$ is redundant in the same way that 2 was redundant in $\mathcal{T}_5$), we claim that no element of $\mathcal{T}_5$ is redundant. Certainly, if $1 \in \mathcal{T}_p$ for any prime p, then it is not redundant, for then we have $Z_p \in \mathfrak{F}(\mathcal{T})$. In particular, Z_7 and Z_{31} apppear in $\mathfrak{F}(\mathcal{T})$. The cyclic group of order 31 is irreducibly represented on a module V of dimension 3 over GF (5), and Z_7 is irreducibly represented on a module W of dimension 6 over GF (5). Hence $[V]\cdot Z_{31}$ and $[W]\cdot Z_7$ both belong to $\mathfrak{F}(\mathcal{T})$, and in particular we have $\min \mathcal{T}_5 = \{1, 3, 6\}$. It is a trivial observation that $\min \mathcal{T}_5$ is neither closed under factorisation nor under multiplication.

(b) We shall show later that the following ranking function Q defines a saturated formation:

$$Q_2 = Q_3 = \{1\} ;$$
$$Q_5 = \{2^i . 3^j : i, j \in \mathbb{N} \cup \{0\}\} ;$$
$$Q_p = \{1\} \quad \text{for } p \geq 7.$$

It is easy to see that this Q is minimal, for $m \in Q_5$, then $Z_{5^m-1} \in \mathfrak{F}(Q)$ since every supersoluble group is contained in $\mathfrak{F}(Q)$, and Z_{5^m-1} is faithfully and irreducibly represented on a module V of dimension m over GF (5). Hence $[V]\cdot Z_{5^m-1} \in \mathfrak{F}(Q)$. Consider the integer $6 \in Q_5$. This has three distinct Q_5-factorisations:

(i) $\{1, 7\}_5$; for $6 = \deg_7 5$.

(ii) $\{3, 3\}_5$; for $6 = 3 \deg_3 5$ and this 5-factorisation is clearly Q-integral.

(iii) $\{2, 31\}_5$; for $6 = 2 . \deg_{31} 5$ and $1 = \deg_2 31 \in \mathcal{R}_{31}$.

In fact, (i), (ii) and (iii) are examples of the three possibilities in any Q-factorisation $\{m_a, m_b\}_p$ of $m \in Q_p$ - either $m_a = 1$, or $q \mid (m_a, m_b)$, or else $q \mid m_a$ and $q \nmid m_b$. However, there is no guarantee that any of these three events need always occur. For example, in (a) there is a unique $\mathcal{S}_5$-factorisation of 3 and of 6 - namely, $\{1, 31\}_5$ for 3 and $\{1, 7\}_5$ for 6. Consider also the following ranking function:

$$\mathfrak{U}_2 = \{3 . 2^i : i \geq 0\};$$
$$\mathfrak{U}_3 = \emptyset;$$
$$\mathfrak{U}_p = \{1\} \quad \text{for } p \geq 5.$$

It will transpire that $\mathfrak{Z}(\mathfrak{U})$ is saturated, and it is easily seen that $\mathfrak{U}$ is minimal. Although $6 \in \mathfrak{U}_2$, there is no cyclic group in $\mathfrak{Z}(\mathfrak{U})$ having a faithful irreducible module of dimension 6 over GF (2). Such a cyclic group G would have order dividing $2^6 - 1 = 63 = 7 . 3^2$. For G to belong to $\mathfrak{Z}(\mathfrak{U})$, it must be a $3'$-group, and so can only be Z_7. However, an irreducible non-trivial Z_7-module has dimension 3 over GF (2), and therefore no such G can exist. Thus if $\{t_a, t_b\}_2$ is a $\mathfrak{U}_2$-factorisation of 6 then $t_a \neq 1$. In fact, $\{2, 7\}_2$ is the unique $\mathfrak{U}_2$-factorisation of 6.

In view of Example III.1.10 (a) and Lemma III.1.9, it seems that we have to rely on p-factorisation to define a composition of two elements of p. The eccentricities demonstrated in III.1.10 (b) force us to look beyond the usual arithmetic notions of product.

III.1.11 Definition. Let m, n $\in \mathcal{S}_p$. The operation $*$ defined by

$$m * n = \{m_a\, n_a\, \deg_{[m_a, m_b]}\, p : \{m_a, m_b\}_p \text{ and } \{n_a, n_b\}_p \text{ are } \mathcal{S}_p\text{-factorisations of } m \text{ and } n \text{ respectively}\}$$

is called the $\mathcal{S}_p$-product of m and n.

III.1.12 Lemma Let $\{m_a, m_b\}_p$, $\{n_a, n_b\}_p$ be $\mathcal{S}_p$-factorisations of m and n respectively with m, $n \in \mathcal{S}_p$. Then $\{m_a . n_a, [m_b, n_b]\}_p$ is an $\mathcal{S}_p$-factorisation of $m_a\, n_a\, \deg_{[m_b, n_b]} p$.

Proof If $q \mid m_a\, n_a$ but $q \nmid [m_b, n_b]$, then because q divides m_a or n_a there is a prime r dividing $[m_b, n_b]$ such that $\deg_q r \in \mathcal{R}_r$.

III.1.13 Proposition If m and n belong to $\mathcal{R}_p$ then $m * n \in \mathcal{R}_p$.

Proof Let $\{m_a, m_b\}_p$ be an $\mathcal{R}_p$-factorisation of m and $\{n_a, n_b\}_p$ an $\mathcal{R}_p$-factorisation of n. Let $q_1^{b_1} \dots q_k^{b_k}\, r_1 \dots r_l$ be the prime decomposition of $m_a\, n_a$, where $r_1 \dots r_l = (m_a\, n_a, [m_b, n_b])$. We show that the properties (a) - (d) of III.1.5 are possessed by $t = m_a\, n_a\, \deg_{[m_b, n_b]} p$. It is clear that $\{m_a n_a, [m_b, n_b]\}_p$ is a p-factorisation of t, and so (a) of III.1.5 holds immediately. If $1 \leq j \leq l$ then because either $r_j \mid m_a$ or $r_j \mid n_a$, it follows from III.1.8 (a) there is an extraspecial group of order r_j^3 in f (p). Hence f (p) contains all extraspecial r_j-groups, proving that (d) of III.1.5 holds. Now let $i \in \{1, \dots, k\}$. Then, without loss of generality, we may assume that $q_i \mid m_a$. Since $\{m_a, m_b\}_p$ is an $\mathcal{R}_p$-factorisation of m, there is a prime s_i with $s_i \mid m_b$ and $E(q_i, s_i) \in f(p)$ by III.1.5. This immediately proves that (b) of III.1.5 holds for t, since $s_i \mid [m_b, n_b]$. Lastly, again by III.1.5, we have $Z_{m_b} \times Z_{n_b} \in f(p)$.

Clearly,

$$Z_{[m_b, n_b]} \in Q\,(Z_{m_b} \times Z_{n_b}) \subseteq f\,(p).$$

Since t has all the properties (a) - (d) of III.1.5, we deduce that $t \in \mathcal{R}_p$. q.e.d.

We define sets $\mathcal{S}_p^*$ and $\mathcal{S}_{p*}$ associated with $\mathcal{S}_p$ that are of especial interest to us.

III.1.14 Definition

$$\mathcal{S}_p^* = \{\, n \in \mathbb{N} : n \text{ is } \mathcal{S}\text{-potent and } n \mid m \text{ for some } m \in \mathcal{S}_p \,\}.$$

$$\mathcal{S}_{p*} = \{\, q \in \mathbb{P} : \mathcal{S}_q \neq \emptyset \text{ and } \deg_q p \mid m \text{ for some } m \in \mathcal{S}_p \,\}.$$

We define the amalgamation of $\mathcal{S}_p^*$ with $\mathcal{S}_{p*}$, denoted by $A\,(\mathcal{S}_p^*, \mathcal{S}_{p*})$, to be the set

$\{\, m_a \deg_{m_b} p : m_a \in \mathcal{S}_p^*$, m_b is a product of distinct primes in $\mathcal{S}_{p*}$, and the p-factorisation $\{m_a, m_b\}_p$ of $m_a \deg_{m_b} p$ is $\mathcal{S}_p$-integrated $\}$.

III.1.15 Proposition For all $p \in \mathbb{P}$;

(a) $\mathcal{R}_p^*$ is closed under the usual multiplication and taking divisors. That is

(i) if $\{m, n\} \subseteq \mathcal{R}_p^*$ then $m\,n \in \mathcal{R}_p^*$, and

(ii) if $m \mid n \in \mathcal{R}_p^*$ then $m \in \mathcal{R}_p^*$.

(b) If $p \neq q \in \mathcal{R}_p^* \cap \mathbb{P}$ then $q \in \mathcal{R}_{p*}$.

(c) If $q \in \mathcal{R}_{p*}$ then $\deg_q p \in \mathcal{R}_p$ and $Z_q \in f\,(p)$.

Proof (a) It is clear from the definition that $\mathcal{R}_p^*$ is closed under taking divisors. Let n and n' be elements of $\mathcal{R}_p^*$ and let m, m' be corresponding elements of $\mathcal{R}_p^*$ - i.e. $n \mid m$ and $n' \mid m'$.

Let $n\,n' = p^a q_1^{b_1} \ldots q_k^{b_k}$ be the prime factorisation of $n\,n'$, where $a \in \mathbb{N} \cup \{0\}$ and $b_i \in \mathbb{N}$ for $1 \leq i \leq k$. For $1 \leq i \leq k$ we have either $q_i \mid m$ or $q_i \mid m'$ and so, by III.1.8 (a) there is an extraspecial group of order q_i^3 in $f(p)$. By taking a central product of b_i copies of such an extraspecial group, there is an extraspecial group E_i of order $q_i^{2b_i+1}$ in $f(p)$. If $a \neq 0$ then again either $p \mid m$ or $p \mid m'$ and so by III.1.8 (b) there is an $\mathfrak{R}$-potent prime $r \neq p$ such that $E(p, r) \in f(p)$. If

$$G = \mathop{\times}_{i=1}^{a} E(p, r) \times \mathop{\times}_{i=1}^{k} E_i,$$

then $G \in f(p)$ and the usual argument involving I.4.12 and I.4.13 ensures the existence of a faithful irreducible G-module of dimension $d = p^a \cdot \left(\prod_{i=1}^{k} q_i^{b_i}\right) \deg_{q_1 \ldots q_k} p.e$ where $e \mid \deg_r p$. We must therefore have $d \in \mathfrak{R}_p$. Since $n n' \mid d$, we have $n n' \in \mathfrak{R}_p^*$, as desired.

(b) If $p \neq q \in \mathfrak{R}_p^* \cap \mathbb{P}$ then by III.1.8 (a) there is an extraspecial group E of order q^3 in $f(p)$. Hence $Z_q \in Q(E) \subseteq f(p)$, giving the result.

(c) This follows immediately from the definition and Lemma III.1.9.

q.e.d.

III.1.16 Lemma If $\mathfrak{R}$ is a full ranking function (see II.4.1 (a)) then $\mathfrak{R}_p = \mathfrak{R}_p^*$.

Proof Clearly $\mathfrak{R}_p \subseteq \mathfrak{R}_p^*$, since every prime dividing $m \in \mathfrak{R}_p$ is $\mathfrak{R}$-potent. Let $n \in \mathfrak{R}_p^*$ and take $m \in \mathfrak{R}_p$ with $n \mid m$. Set $m = n n'$. Since $\mathfrak{R}$ is full it follows that $\mathfrak{J}(\mathfrak{R})$ contains the class of supersoluble groups - in particular, $Z_{p^m-1} \in \mathfrak{J}(\mathfrak{R})$. Since

Z_{p^m-1} is faithfully and irreducibly represented on a module of dimension m over GF (p), we have $Z_{p^m-1} \in f$ (p). Now,

$p^m - 1 = (p^n - 1)(p^{n(m-1)} + p^{n(m-2)} + \ldots + p^n + 1)$ and so

$Z_{p^n-1} \in Q(Z_{p^m-1}) \subseteq f$ (p). Since Z_{p^n-1} is irreducibly represented on a module of dimension n, we deduce that $n \in \mathfrak{R}_p$.

q.e.d.

<u>III.1.17 Corollary</u> (Heineken [25], Lemma 1). If $\mathfrak{R}$ is full then $\mathfrak{R}_p$ is closed under factorisation and multiplication for all primes p.

<u>Proof</u> This is a consequence of III.1.16 and III.1.15 (a).

q.e.d.

We are now able to state four essential properties of $\mathfrak{R}_p$ for each prime p. We designate them as follows for our general ranking function .

R F 1: If n is $\mathcal{S}_p$-integrated and $n \mid m \in \mathcal{S}_p$, then $n \in \mathcal{S}_p$.

R F 2: If $\{m, n\} \subseteq \mathcal{S}_p$, then $m * n \subseteq \mathcal{S}_p$.

R F 3: $\mathcal{S}_p^*$ is multiplication and factorisation closed.

R F 4: $\mathcal{S}_p^* \cap \mathbb{P} \setminus \{p\} \subseteq \mathcal{S}_{p*}$.

In an attempt to indicate the effect of the properties which we accumulate for $\mathfrak{R}$, we shall introduce an example which will be referred to several times in this chapter.

<u>Example (*)</u> Let $\mathfrak{U}$ be a ranking function. We wish that $\mathfrak{J}(\mathfrak{U})$ should be saturated, and that

$2 \in \mathfrak{U}_2$;

$2 \in \mathfrak{U}_3$;

$1 \in \mathfrak{U}_{11}$;

and $\mathfrak{U}_p = \emptyset$ for all $p \in \mathbb{P} \setminus \{2, 3, 11\}$. As we add conditions on $\mathfrak{R}$, we shall see what effects these conditions have on $\mathfrak{U}$. By III.1.1 we must have $1 \in \mathfrak{U}_2 \cap \mathfrak{U}_3$. Notice that $1 = \deg_2 3$. By R F 2, then, we have $2^i \deg_2 3 \in \mathfrak{U}_3$ for $i \geqslant 0$, since $\{2, 2\}_3$ is a $\mathfrak{U}_3$-factorisation of $2 \in \mathfrak{U}_3$. On the other hand,

$2 = \deg_3 2 \in \mathfrak{U}_2$ and $2 \in \mathfrak{U}_2^*$. Hence $2^i \in \mathfrak{U}_2^*$ by R F 3.

Recapping, we have

$$2 \in \mathfrak{U}_2, \ \{2^i : i \geqslant 0\} \subseteq \mathfrak{U}_2^* \text{ and } 3 \in \mathfrak{U}_{2*}$$

$$\{2^i : i \geqslant 0\} \subseteq \mathfrak{U}_3 \cap \mathfrak{U}_3^* \text{ and } 2 \in \mathfrak{U}_3.$$

Evidently, to make sense of $\mathfrak{U}_2$ (as it stands) we need to know the relationship between $\mathfrak{R}_p^*$ and $\mathfrak{R}_{p*}$.

<u>III.1.18 Proposition</u> $A(\mathfrak{R}_p^*, \mathfrak{R}_{p*}) = \mathfrak{R}_p$.

<u>Proof</u> It is obvious from III.1.5 (a) and (b) that $\mathfrak{R}_p \subseteq A(\mathfrak{R}_p^*, \mathfrak{R}_{p*})$. Let $m_a \in \mathfrak{R}_p^*$ and let m_b be a product of distinct primes from $\mathfrak{R}_{p*}$ such that $\{m_a, m_b\}_p$ is an $\mathfrak{R}_p$-factorisation of $m = m_a . \deg_{m_b} p$.

Let $m_a = q_1^{b_1} \ldots q_k^{b_k} p^a$ be a prime factorisation of m_a with $b_i \in \mathbb{N}$ for $1 \leqslant i \leqslant k$, and $a \in \mathbb{N} \cup \{0\}$. If $1 \leqslant i \leqslant k$, then there exists $m_i \in \mathfrak{R}_p$ such that $q_i \mid m_i$. By III.1.8 (a) there is an extraspecial group of order q_i^3 in $f(p)$, and so by taking a certain central product of b_i copies of such an extraspecial group, we obtain an extraspecial group E_i of order $q_i^{2b_i+1}$ with $E_i \in f(p)$. If $a \neq 0$, then $p \mid m \in \mathfrak{R}_p$. By III.1.8 (b) we then see that there is an $\mathfrak{R}$-potent prime r with $E(p, r) \in f(p)$.

Let $G = \mathop{\times}\limits_{i=1}^{a} E(p, r) \times \mathop{\times}\limits_{j=1}^{k} E_j \in f(p)$.

Lastly, $Z_{m_b} \in f(p)$ by III.1.5 (c). Thus

$G \times Z_{m_b} \in f(p)$. Using I.4.12 we see that $G \times Z_{m_b}$ is irreducibly

represented on a module of dimension $m' = p^a \cdot \prod_{i=1}^{k} q_i^{b_i} \cdot \deg_{[m_a, r, m_b]} p$

over $GF(p)$. Hence $m' \in \mathcal{R}_p$. Since $m \mid m'$ it now follows from III.1.9 that $m \in \mathcal{R}_p$, as required.
q.e.d.

__Example (*) (continued)__ In view of III.1.18, we see that

$$\{2^i : i \geq 0\} \subseteq \mathcal{U}_2$$

since $\deg_2 3 = 1 \in \mathcal{U}_3$, whence the integer $2^j . \deg_3 2$ is $\mathcal{U}_2$-integrated with $\mathcal{U}_2$-factorisation $\{2^j , 3\}_2$, and therefore belongs to $A(\mathcal{U}_2^* , \mathcal{U}_{2*})$.

The four conditions R F 1, R F 2, R F 3 and R F 4 are still far from being sufficient to ensure that $\mathfrak{F}(\mathcal{R})$ is saturated, as the next two propositions and their application to Example (*) will demonstrate.

__III.1.19 Proposition__ Let p, q and r be $\mathcal{R}$-potent primes such that

(a) $p \neq q \in \mathcal{R}_p^*$, and

(b) $r \in \mathcal{R}_{q*}$.

Then $r \in \mathcal{R}_p^*$.

__Proof__ We note that by III.1.15 (b) and (c), we have

$$Z_q \in f(p) \text{ and } \deg_q p \in \mathcal{R}_p.$$

Hence the groups $G = E(r , q)$ and $H = E(q , p)$ both belong to $\mathfrak{F}(\mathcal{R})$. Let $T \in \mathrm{Syl}_r(G)$ (and notice that $T \in f(q)$).

We set $W = N \wr_T G$. Since $T \in \mathfrak{f}(q)$, it follows from II.1.19 (b) that the q-chief factors of W are all $\mathfrak{R}$-admissible. Again, by II.1.19, the group W is primitive with unique minimal normal

subgroup, N say, of order $p^{q^d e}$ where $d = \deg_r q$ (and so $|G : T| = q^d$) and $e = \deg_q p$. Since from (a) we have $q \in \mathfrak{R}_p^*$, and hence $q \in \mathfrak{R}_{p*}$ by III.1.15 (b), it follows from III.1.18 that $q^d e \in \mathfrak{R}_p$. So N may be considered as an $\mathfrak{R}$-admissible p-chief factor of W. Since the remaining chief factor of r is central of $\mathfrak{R}$-potent order r, we see that $W \in \mathfrak{J}(\mathfrak{R})$.

Let M be a complement of N in W. Then $M \in \mathfrak{f}(p)$. Further, we now see that $G \in Q(M) \subseteq \mathfrak{f}(p)$. By I.4.10, a faithful, irreducible G-module has dimension $[r, \deg_q p]$ over GF (p). Since we must now have $[r, \deg_q p] \in \mathfrak{R}_p$, it follows that

$$r \in \mathfrak{R}_p^*,$$

as $r \mid [r, \deg_q p]$.
q.e.d.

We annotate the property described in III.1.19 as follows:

R F 5: if p, q and r are $\mathcal{S}$-potent primes such that

(a) $p \neq q \in \mathcal{S}_p^*$, and

(b) $r \in \mathcal{S}_{q*}$,

then $r \in \mathcal{S}_p^*$.

We remark that this is equivalent in $\mathcal{S}$ (via the relevant definitions) to the following:

if p, q and r are $\mathcal{S}$-potent primes such that

(a) $p \neq q \mid m \in \mathcal{S}_p$, and

(b) $r \mid q^l - 1$ for some $l \in \mathcal{S}_q$,

then $r \mid n$ for some $n \in \mathcal{S}_p$.

<u>Example (*) (continued)</u> At present we have

$$\{2^i : i \geqslant 0\} \subseteq \mathcal{U}_2 \cap \mathcal{U}_2^* ;$$

$$3 \in \mathcal{U}_2^* ;$$

$$\{2^i : i \geqslant 0\} \subseteq \mathcal{U}_3 \cap \mathcal{U}_3^* ;$$

$$2 \in \mathcal{U}_{3*} ;$$

Observe that $3 \nmid 2 \in \mathcal{U}_3^*$ and $3 \in \mathcal{U}_{2*}$. Since $\mathcal{U}$ must have property R F 5, we deduce that 3 must belong to $\mathcal{U}_3^*$. Since $\deg_3 2 = 2 \in \mathcal{U}_2$, we infer that the integer $3^i \deg_2 3$ is $\mathcal{U}_3$-integrated and belongs to $A(\mathcal{U}_3^*, \mathcal{U}_{3*})$. Hence $\{3^i : i \geqslant 0\} \subseteq \mathcal{U}_3$. Since $\mathcal{U}$ has property R F 2, we now deduce that

$$\{2^i 3^j : i, j \geqslant 0\} \subseteq \mathcal{U}_3 \cap \mathcal{U}_3^* .$$

Brief inspection of the resultant function reveals that no further application of R F 1 - R F 5 is possible.

In a similar vein to III.1.19, we have

<u>III.1.20 Proposition</u> Let p, q and r be $\mathcal{R}$-potent primes such that

(a) $\{r, q\} \subseteq \mathcal{R}_{p*}$;

(b) $p \in \mathcal{R}_{q*}$;

(c) $p \in \mathcal{R}_p^*$; and

(d) $r \in \mathcal{R}_q^*$.

Then $r \in \mathcal{R}_p^*$.

<u>Proof</u> By (a) and III.1.15 (c) we have $Z_r \in$ [illegible], and so $\deg_r p \in \mathcal{R}_p$. Since $r \in \mathcal{R}_q^*$ and $p \in \mathcal{R}_{q*}$ we get $r.\deg_p q \in A(\mathcal{R}_q^*, \mathcal{R}_{q*})$, and so by III.1.18 we have

(i) $r.\deg_p q \in \mathcal{R}_q$.

Similarly, from (b) we obtain $\deg_p q \in \mathcal{R}_q$, and so since $p \in \mathcal{R}_p^*$, (and hence by III.1.15 (a), $p^i \in \mathcal{R}_p^*$ for $i \geqslant 0$) and $q \in \mathcal{R}_{p*}$, it follows from III.1.18 that

(2) $p^i.\deg_q p \in \mathfrak{R}_p$ for $i \geq 0$.

Let $W = E(p, q) \wr Z_r$. By II.1.19 the group W is primitive. Let N denote the minimal normal subgroup of W and let H be a complement of N in W. By II.1.19, with $d = \deg_p q$, we have $|N| = q^{rd}$. Therefore N is an $\mathfrak{R}$-admissible chief factor of W by (1). The p-chief factors of W all induce groups of automorphisms in W which are isomorphic to quotients of $Z_r \in f(p)$. Hence the p-chief factors of W are all $\mathfrak{R}$-admissible. The remaining chief factor is central of $\mathfrak{R}$-potent order r, and so all chief factors of W are $\mathfrak{R}$-admissible. Consequently, $W \in \mathfrak{F}(\mathfrak{R})$.

By II.1.19 (c), there is a faithful, irreducible W-module V over GF (p) of dimension $p^s.\deg_q p$ for some s.

It is now a consequence of (2) that $[V] \cdot W \in \mathfrak{F}(\mathfrak{R})$, whence $W \in f(p)$. Now appealing to II.1.18, there is an integer n such that the group $W^* = [N \times \overset{n}{\ldots} \times N] \cdot H$ is faithfully and irreducibly represented on a module U of dimension m over GF (p) with $|H| \mid m$. As f is an integrated local definition of $\mathfrak{F}(\mathfrak{R})$, we now get $[U] \cdot W \in \mathfrak{F}(\mathfrak{R})$, and we must therefore have $m \in \mathfrak{R}_p$. Because $r \mid |H|$ and, $|H| \mid m$, we now deduce that $r \in \mathfrak{R}_p^*$.

q.e.d.

Again, for reference, we define the following property:

R F 6: If p, q and r are $\mathfrak{S}$-potent primes such that

(a) $\{r, q\} \subseteq \mathfrak{S}_{p*}$;

(b) $p \in \mathfrak{S}_{q*}$;

(c) $p \in \mathfrak{S}_p^*$;

(d) $r \in \mathfrak{S}_q^*$,

then $r \in \mathfrak{S}_p^*$.

The property R F 6 may be interpreted in a manner similar to R F 5: $\mathcal{S}$ has R F 6 if and only if:

If p, q and r are $\mathcal{S}$-potent primes such that

(a) $r \mid p^a - 1$ for some $a \in \mathcal{S}_p$;

(b) $p \mid q^b - 1$ for some $b \in \mathcal{S}_q$;

(c) $q \mid p^c - 1$ for some $c \in \mathcal{S}_p$;

(d) $p \mid m$ for some $m \in \mathcal{S}_p$; and

(e) $r \mid n$ for some $n \in \mathcal{S}_q$,

then $r \mid l$ for some $l \in \mathcal{S}_p$.

<u>Example (*) (continued)</u> So far we have:

$$\{2^i : i \geq 0\} \in \mathcal{U}_2 \cap \mathcal{U}_2^*$$
$$3 \in \mathcal{U}_{2*}$$
$$\{2^i 3^j : i, j \geq 0\} \subseteq \mathcal{U}_3 \cap \mathcal{U}_3^*$$
$$2 \in \mathcal{U}_{3*}$$

Observe that $2 \in \mathcal{U}_2^* \cap \mathcal{U}_{3*}$ and $3 \in \mathcal{U}_{2*}$. We see now that if we set $r = q = 3$ and $p = 2$, then the hypotheses of R F 6 are satisfied. Now, we require that $\mathfrak{F}(\mathcal{U})$ is saturated and therefore, by III.1.20, $\mathcal{U}$ must have property R F 6. Thus we must have $3 \in \mathcal{U}_2^*$. It is now easy to see that $\{2^i 3^j : i, j \geq 0\} \subseteq \mathcal{U}_2$. One can check quite readily that R F 6 now holds in $\mathcal{U}$. We shall see later that $\mathcal{U}$ with

$$\mathcal{U}_2 = \mathcal{U}_3 = \{2^i 3^j : i, j \geq 0\};$$
$$\mathcal{U}_{11} = \{1\}; \text{ and}$$
$$\mathcal{U}_p = \emptyset \text{ for } p \in \mathbb{P} \setminus \{2, 3, 11\},$$

defines a saturated formation $\mathfrak{F}(\mathcal{U})$. We have also seen that none of the conditions R F 1, R F 2, R F 3, R F 4, R F 5 and R F 6 nor the condition that $\mathcal{U}$ should be integrated, are redundant in a list of necessary and sufficient conditions determining a saturated formation, for each has contributed to the final form of $\mathfrak{U}$ above.

In reviewing the definitions of R F 1 – R F 6 and of integrated elements of $\mathcal{R}$, we observe that they may all be defined using $\mathcal{R}_p^*$ and $\mathcal{R}_{p*}$ with the exception of R F 1 and R F 2. In view of the fact that $\mathcal{R}_p^*$ is well-behaved in comparison to $\mathcal{R}_p$ itself, it would obviously be desirable to remedy this. The resultant lemma simplifies matters.

We shall say that $\mathcal{S}$ has R F k for $k \in \{1, \dots, 6\}$ if R F k always holds in $\mathcal{S}$.

<u>III.1.21 Lemma</u> The following statements are equivalent.

(<u>a</u>) $\mathcal{S}$ has R F 1, R F 2, R F 3 and is integrated.

(<u>b</u>) $\mathcal{S}$ has the following properties:

R F 1^*: $\mathcal{S}_p^*$ is divisor and multiplication closed for all p.

R F 2^*: $A(\mathcal{S}_p^*, \mathcal{S}_{p*}) = \mathcal{S}_p$ for all $p \in \mathbb{P}$.

<u>Proof</u>: <u>(a)$\Rightarrow$(b)</u>. We first show that R F 2^* holds. Certainly, $\mathcal{S}_p \subseteq A(\mathcal{S}_p^*, \mathcal{S}_{p*})$ since $\mathcal{S}$ is integrated. Choose m minimally in $A(\mathcal{S}_p^*, \mathcal{S}_{p*}) \setminus \mathcal{S}_p$, and let $\{m_a, m_b\}_p$ be a $\mathcal{S}_p$-factorisation of m. Since $\mathcal{S}$ has R F 1 we have that $m_a \neq 1$. For if $q \mid m_b$ then $\deg_q p \mid n \in \mathcal{S}_p$ and hence $\deg_q p \in \mathcal{S}_p$. This argument also shows that $\deg_{m_b} p \in \mathcal{S}_p$ since $\mathcal{S}$ has R F 2. We claim that

m_a and m_b may both be chosen to be primes. Suppose firstly that $m_a = m'_a m''_a$ where $1 \neq m'_a, m''_a$. Then

$$m = m_a \deg_{m_b} p = m'_a m''_a \deg_{m_b} p$$
$$\in (m'_a \deg_{m_b} p) * (m''_a \deg_{m_b} p).$$

Clearly $m'_a \deg_{m_b} p$ and $m''_a \deg_{m_b} p$ both belong to $A(\mathcal{S}^*_p, \mathcal{S}_{p*})$ (it is obvious that $\mathcal{S}_{p*}$ is divisor closed) and are smaller than m, so belong to $\mathcal{S}_p$. The fact that $\mathcal{S}$ has R F 2 would yield $m \in \mathcal{S}_p$, against assumption.

Therefore m_a is a prime. Next suppose that m_b is not a prime. Since m is integrated, there is a prime r such that $r \mid m_b$ and either $r = m_a$ or $\deg_{m_a} r \in \mathcal{S}_p$. Suppose further that $\deg_r p < \deg_{m_b} p$. In any case, $m_a \deg_r p \in A(\mathcal{S}^*_p, \mathcal{S}_{p*})$. We have already observed that $\deg_{m_b} p \in A(\mathcal{S}^*_p, \mathcal{S}_{p*})$, and by supposition, both $m_a \deg_r p$ and $\deg_{m_b} p$ are smaller than m, so $\{m_a \deg_r p, \deg_{m_b} p\} \subseteq \mathcal{S}_p$. But since $\mathcal{S}$ has R F 2, we now infer that

$$m = m_a \deg_{[r, m_b]} p = m_a \deg_{m_b} p \in m_a \deg_r p * \deg_{m_b} p \subseteq \mathcal{S}_p,$$

against our original assumption. Therefore either $\deg_r p = \deg_{m_b} p$ or else m_b is a prime. We may anyway assume that m_b is a prime by choice of r.

Since $m_a \in \mathcal{S}^*_p$, there is an element k of $\mathcal{S}_p$ such that $m_a \mid k$. Let $\{k_a, k_b\}_p$ be an $\mathcal{S}_p$-factorisation of k. Recall that we have $\deg_{m_b} p \in \mathcal{S}_p$. Then $l = k_a \deg_{[k_b, m_b]} p \in k * \deg_{m_b} p \subseteq \mathcal{S}_p$ by R F 2. Certainly $[m_a, \deg_{m_b} p] \mid l$. If, in fact, $m \mid l$ then we are finished by applying R F 1. Otherwise, let $l_a = k_a$, $l_b = [k_b, m_b]$, and suppose that $m \nmid l$. By III.1.12 the factorisation $\{l_a, l_b\}_p$ of l is an $\mathcal{S}_p$-factorisation. Let $m_a^d \nmid \deg_{m_b} p$. Since R F 3 holds for $\mathcal{S}$, there is an element

Clearly $m'_a \deg_{m_b} p$ and $m''_a \deg_{m_b} p$ both belong to $A(\mathcal{S}^*_p, \mathcal{S}_{p^*})$ (it is obvious that $\mathcal{S}_{p*}$ is divisor closed) and are smaller than m, so belong to $\mathcal{S}_p$. The fact that $\mathcal{S}$ has R F 2 would yield $m \in \mathcal{S}_p$, against assumption.

Therefore m_a is a prime. Next suppose that m_b is not a prime. Since m is integrated, there is a prime r such that $r \mid m_b$ and either $r = m_a$ or $\deg_{m_a} r \in \mathcal{S}_p$. Suppose further that $\deg_r p < \deg_{m_b} p$. In any case, $m_a \deg_r p \in A(\mathcal{S}^*_p, \mathcal{S}_{p*})$. We have already observed that $\deg_{m_b} p \in A(\mathcal{S}^*_p, \mathcal{S}_{p*})$, and by supposition, both $m_a \deg_r p$ and $\deg_{m_b} p$ are smaller than m, so $\{m_a \deg_r p, \deg_{m_b} p\} \subseteq \mathcal{S}_p$. But since $\mathcal{S}$ has R F 2, we now infer that

$$m = m_a \deg_{[r, m_b]} p = m_a \deg_{m_b} p \in m_a \deg_r p * \deg_{m_b} p \subseteq \mathcal{S}_p,$$

against our original assumption. Therefore either $\deg_r p = \deg_{m_b} p$ or else m_b is a prime. We may anyway assume that m_b is a prime by choice of r.

Since $m_a \in \mathcal{S}^*_p$, there is an element k of $\mathcal{S}_p$ such that $m_a \mid k$. Let $\{k_a, k_b\}_p$ be an $\mathcal{S}_p$-factorisation of k. Recall that we have $\deg_{m_b} p \in \mathcal{S}_p$. Then $l = k_a \deg_{[k_b, m_b]} p \in k * \deg_{m_b} p \subseteq \mathcal{S}_p$ by R F 2. Certainly $[m_a, \deg_{m_b} p] \mid l$. If, in fact, $m \mid l$ then we are finished by applying R F 1. Otherwise, let $l_a = k_a$, $l_b = [k_b, m_b]$, and suppose that $m \nmid l$. By III.1.12 the factorisation $\{l_a, l_b\}_p$ of l is an $\mathcal{S}_p$-factorisation. Let $m_a^d \Vert \deg_{m_b} p$. Since R F 3 holds for $\mathcal{S}$, there is an element

j of $\mathcal{S}_p$ with $m_a^{d+1} \mid j$. Let $\{j_a , j_b\}_p$ be an $\mathcal{S}_p$ factorisation of j. Then $l' = j_a l_a \deg_{[j_b, l_b]} p \in j * l \subseteq \mathcal{S}_p$. Since

$m_b \mid [j_b , l_b]$ and $m_a^{d+1} \mid j$ we must have $m \mid l'$. We again apply

R F 1 to get a contradiction. Hence no such m exists, and so $A(\mathcal{S}_p^* , \mathcal{S}_{p*}) = \mathcal{S}_p$. Also, R F 1^* certainly holds, since RF3 holds.

<u>(b)$\Rightarrow$(a)</u> This direction is obvious.

q.e.d.

The condition RF3 is not redundant in III.1.21 (a), as the next example shows:

<u>III.1.22 Example</u> Let $\mathcal{S}$ be defined by

$$\mathcal{S}_2 = \{1\} , \mathcal{S}_3 = \mathcal{S}_{31} = \emptyset , \mathcal{S}_5 = \{1 , 6\} \text{ and}$$

$\mathcal{S}_p = \{1\}$ for $p \in \mathbb{P} \setminus \{2 , 3 , 5 , 31\}$.

Brief consideration of the alternatives shows that $\{1, 7\}_5$ is the unique $\mathcal{S}_5$-factorisation of 6. It is therefore clear that $\mathcal{S}$ has R F 1 and R F 2 as well as being integrated. However, $\mathcal{S}_5^* = \{1 , 2\}$ and $\mathcal{S}_{5*} = \{2 , 7\}$ and we see that

$$2 \deg_7 5 = 12 \in A(\mathcal{S}_5^* , \mathcal{S}_{5*}) \setminus \mathcal{S}_5$$

and $\mathcal{S}_5^*$ is not multiplicatively closed.

Since we have shown that $\mathcal{R}$ has R F 1, R F 2 and R F 3, we may assume also that $\mathcal{R}$ has R F 1^* and R F 2^*.

<u>III.1.23 Lemma</u> (Heineken [25]) Let p and q be $\mathcal{S}$-potent primes and suppose that $\mathcal{S}$ is an integrated ranking function with R F 1, R F 2, R F 3, R F 4 and R F 5.

(a) If $p \neq 2$ and $\mathcal{S}_2 \neq \emptyset$ then $2 \in \mathcal{S}_{p*}$

(b) If $p \neq q \in \mathcal{S}_p^*$ and $\mathcal{S}_2 \neq \emptyset$ then $2 \in \mathcal{S}_p^*$.

(c) If $p \neq q \in \mathcal{S}_p^*$ then $\mathcal{S}_q^* \subseteq \mathcal{S}_p^*$ and $\mathcal{S}_{q*} \setminus \{p\} \subseteq \mathcal{S}_{p*}$.

Proof (a) Since p is odd it follows that $1 = \deg_2 p$, and so $2 \in \mathcal{S}_{p*}$.

(b) If $q = 2$ then we are done. Otherwise, since q is $\mathcal{S}$-potent we have $2 \in \mathcal{S}_{q*}$ by (a) and $2 \in \mathcal{S}_p^*$ since $\mathcal{S}$ has R F 5.

(c) Since $\mathcal{S}_q^*$ and $\mathcal{S}_p^*$ both satisfy R F 1^* by III.1.21, in order to prove that $\mathcal{S}_q^* \subseteq \mathcal{S}_p^*$, it is sufficient to show that if $r \in \mathcal{S}_q^* \cap \mathbb{P}$ then $r \in \mathcal{S}_p^* \cap \mathbb{P}$. We can then deduce that $\mathcal{S}_q^* \subseteq \mathcal{S}_p^*$. Of course, we may assume that $p \neq q$. Let $q \neq r \in \mathcal{S}_q^*$. Since $\mathcal{S}$ has R F 4, we have $r \in \mathcal{S}_q^*$. We can now use R F 5 to infer that $r \in \mathcal{S}_p^*$.

Notice that applying R F 5 in fact gives us $\mathcal{S}_{q*} \subseteq \mathcal{S}_{p*}$. Thus appealing to R F 4 we have $\mathcal{S}_{q*} \setminus \{p\} \subseteq \mathcal{S}_{p*}$. q.e.d.

III.1.24 Corollary If q and p are $\mathcal{S}$-potent primes, where $\mathcal{S}$ is as in III.1.23, such that $p \neq q \in \mathcal{S}_p^*$ and either $p \notin \mathcal{S}_{q*}$ or $\mathcal{S}$ is full, then $\mathcal{S}_q \subseteq \mathcal{S}_p$.

Proof Suppose first that $p \notin \mathcal{S}_{q*}$. Then by III.1.23 (c) we have $\mathcal{S}_q^* \subseteq \mathcal{S}_p^*$ and $\mathcal{S}_{q*} \subseteq \mathcal{S}_{p*}$. By III.1.2 we then have

$$\mathcal{S}_q = \Lambda(\mathcal{S}_q^*, \mathcal{S}_{q*}) \subseteq \Lambda(\mathcal{S}_p^*, \mathcal{S}_{p*}) = \mathcal{S}_p.$$

If $\mathcal{S}$ has full support, then from III.1.16 and III.1.23 (c) we get

$$\mathcal{S}_q = \mathcal{S}_q^* \subseteq \mathcal{S}_p^* = \mathcal{S}_p. \quad \text{q.e.d.}$$

In his paper, Heineken [25] states and proves several more results in the nature of III.1.23 and III.1.24 for full ranking functions having R F 1^*, R F 5 and R F 6 (see [25], Corollary 3, Corollary 4, Corollary 5 and Theorem 3) which serve to throw illumination onto the behaviour of such ranking functions. Unfortunately, the most fundamental of these, Corollary 3 (b) of [25], is false in our more general setting, invalidating those which follow it. We shall state this particular result and give a counter-example of a non-full ranking function not having the corresponding property.

<u>III.1.25 Lemma</u> (Heineken [25], Corollary 3 (b)). Let $\mathcal{S}$ be a full ranking function having R F 1^*, R F 5 and R F 6. Suppose that $\{p_1, \dots, p_k : k \geq 2\}$ is a set of distinct primes such that

(i) $p_i \in \mathcal{S}_{p_{i+1}}$, $1 \leq i \leq k-1$, and

(ii) $p_k \in \mathcal{S}_{p_1}$.

Then $\mathcal{S}_{p_i} = \mathbb{N}$ for $1 \leq i \leq k$.

One would hope for a generalisation for an arbitrary integrated ranking function $\mathcal{S}$ having R F 1 , ... , R F 6 to be of the following form.

Let $p_1, \dots, p_k$, $k \geq 2$ be distinct $\mathcal{S}$-potent primes such that

(i) $p_i \in \mathcal{S}^*_{p_{i+1}}$, $1 \leq i \leq k-1$, and

(ii) $p_k \in \mathcal{S}^*_{p_1}$.

Then $\mathcal{S}^*_{p_i} \cap \mathbb{P} = \{p \in \mathbb{P} : \mathcal{S}_p \neq \emptyset\}$, the <u>support</u> of $\mathcal{S}$. By III.1.23 (c) we certainly have $\mathcal{S}^*_{p_1} = \dots = \mathcal{S}^*_{p_k}$, but the above conjecture is false.

<u>III.1.26 Example</u> Let $\mathfrak{U}$ be defined by

$$\mathfrak{U}_2 = \mathfrak{U}_3 = \{2^i 3^j : 0 \leq i, j\};$$

$$\mathfrak{U}_{11} = \{1\};$$

illumination onto the behaviour of such ranking functions. Unfortunately, the most fundamental of these, Corollary 3 (b) of [25], is false in our more general setting, invalidating those which follow it. We shall state this particular result and give a counter-example of a non-full ranking function not having the corresponding property.

III.1.25 Lemma (Heineken [25], Corollary 3 (b)). Let $\mathcal{S}$ be a full ranking function having R F 1*, R F 5 and R F 6. Suppose that $\{ p_1 , \dots , p_k : k \geq 2 \}$ is a set of distinct primes such that

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One would hope for a generalisation for an arbitrary integrated ranking function $\mathcal{S}$ having R F 1 , ... , R F 6 to be of the following form.

Let $p_1 , \dots , p_k$, $k \geq 2$ be distinct $\mathcal{S}$-potent primes such that

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Then $\mathcal{S}^*_{p_i} \cap \mathbb{P} = \{ p \in \mathbb{P} : \mathcal{S}_p \neq \emptyset \}$, the support of $\mathcal{S}$. By III.1.23 (c) we certainly have $\mathcal{S}^*_{p_1} = \dots = \mathcal{S}^*_{p_k}$, but the above conjecture is false.

III.1.26 Example Let $\mathcal{U}$ be defined by

$$\mathcal{U}_2 = \mathcal{U}_3 = \{ 2^i 3^j : 0 \leq i , j \};$$

$$\mathcal{U}_{11} = \{ 1 \};$$

$$\mathcal{U}_p = \emptyset \text{ for } p \in \mathbb{P} \setminus \{2, 3, 11\}.$$

It is easy to check that $\mathcal{U}$ has R F 1, ... , R F 6 and is integrated. But since $11 \notin \mathcal{U}_2$, the above conjecture fails.

2. Characterisation of ranked saturated formations

In Section 1 we have shown that if $\mathcal{R}$ is a ranking function such that $\mathfrak{F}(\mathcal{R})$ is saturated, then $\mathcal{R}$ has the following properties:

(1) $\mathcal{R}$ is integrated (see III.1.5 and III.1.6);

(2) $\mathcal{R}$ has R F 1, R F 2, R F 3, R F 4, R F 5 and R F 6 (see III.1.9, III.1.13, III.1.15 (a) and (b), III.1.19 and III.1.20);

(3) $\mathcal{R}$ has R F 1^*, R F 2^*, R F 4, R F 5 and R F 6 (see III.1.21, III.1.15 (b), III.1.19 and III.1.20).

We shall show that the conditions listed under (1) and (2) or under (1) and (3) are sufficient to ensure that $\mathfrak{F}(\mathcal{R})$ is saturated.

III.2.1 Definition Let $\mathcal{S}$ be a ranking function. Then $\mathfrak{A}(\mathcal{S}_p)$ will denote the class of abelian $\mathcal{S}_{p^*}$-groups (recall: $\mathcal{S}_{p^*} \subseteq \mathbb{P}$) of exponent dividing $p^m - 1$ for $m \in \mathcal{S}_p$.

III.2.2 Lemma Let $\mathcal{S}$ be a ranking function. If $\mathcal{S}$ is integrated and has R F 2 then $\mathfrak{A}(\mathcal{S}_p)$ is a subgroup closed formation.

Proof It is obvious that $\mathfrak{A}(\mathcal{S}_p)$ is subgroup closed and Q-closed. Let $A \in \mathfrak{A}$ have normal subgroups N_1 and N_2 such that $A / N_i \in \mathfrak{A}(\mathcal{S}_p)$ for $i = 1, 2$. In particular, define $m, n \in \mathcal{S}_p$ such that $\exp(A / N_1) \mid p^m - 1$ and $\exp(A / N_2) \mid p^n - 1$. If $l = [m, n]$, then it follows that $A / N_i \in \mathfrak{A}(p^l - 1)$ for $i = 1, 2$ and hence $A / (N_1 \cap N_2) \in \mathfrak{A}(p^l - 1)$ (see II.2.2 for the definition of $\mathfrak{A}(t)$; it is easily seen to be a formation). Thus it suffices to check that 1 divides some element of $\mathcal{S}_p$. Let $\{m_a , m_b\}_p$ and $\{n_a , n_b\}_p$ be $\mathcal{S}_p$-factorisations of m and n respectively. By R F 2, we have $m_a\, n_a \deg_{[m_b, n_b]} p =$

$m_a\, n_a\, [\deg_{m_b} p\, , \deg_{n_b} p] \in \mathcal{S}_p$. Since it is obvious that

$1 \mid m_a\, n_a\, [\deg_{m_b} p\, , \deg_{n_b} p]$ the result follows.
q.e.d.

We can now prove the first main theorem of the chapter.

<u>III.2.3 Theorem</u> Let $\mathfrak{R}$ be a minimal ranking function. For $p \in \mathbb{P}$ set $\pi_p = \mathfrak{R}_p^* \cap \mathbb{P}$. The following statements are equivalent in pairs:

(a) $\mathfrak{Z}(\mathfrak{R})$ is saturated.

(b) $\mathfrak{R}$ has R F 1, R F 2, R F 3, R F 4, R F 5, R F 6 and is integrated.

(c) $\mathfrak{R}$ has R F 1*, R F 2*, R F 4, R F 5 and R F 6.

(d) $\mathfrak{A}(\mathfrak{R}_p)$ is a formation for all $p \in \mathbb{P}$ and $\mathfrak{Z}(\mathfrak{R})$ is locally defined by

$$f(p) = \mathfrak{A}(\mathfrak{R}_p)\mathfrak{S}_{\pi_p}.$$

<u>Proof</u> <u>(a)$\Rightarrow$(b)</u> is done in III.1.5, III.1.9, III.1.13, III.1.15 (a) and (b), III.1.19 and III.1.20.

<u>(b)$\Leftrightarrow$(c)</u> is done in III.1.21.

<u>(d)$\Rightarrow$(a)</u> is a consequence of the Gaschütz-Lubeseder Theorem (III.3.10).

It remains to prove

<u>(c)$\Rightarrow$(d)</u> That $\mathfrak{A}(\mathfrak{R}_p)$ is a formation follows from III.1.21 and

III.2.2. Let $\hat{\mathfrak{F}}(\mathfrak{R})$ be the saturated formation locally defined by $f(p) = \mathfrak{A}(\mathfrak{R}_p)\mathfrak{S}_{\pi_p}$. We first show that $\hat{\mathfrak{F}}(\mathfrak{R}) \subseteq \mathfrak{F}(\mathfrak{R})$. To this end we assume to the contrary, and choose G to be of minimal order in $\hat{\mathfrak{F}}(\mathfrak{R}) \setminus \mathfrak{F}(\mathfrak{R})$. By II.4.13, there is a unique minimal normal subgroup N of G. Since $G/N \in \mathfrak{F}(\mathfrak{R})$, every chief factor of G above N is $\mathfrak{R}$-admissible. Our choice of G means that

(1) $r(N) \notin \mathfrak{R}_p$,

where p is the prime divisor of $|N|$. Suppose that $N \leqslant \Phi(G)$. By II.1.4, N cannot be self-centralising, since it would then be complemented by a maximal subgroup of G. Hence $|[N] \cdot A_G(N)| < |G|$, and so by II.3.8 (b) we have $N.A_G(N) \in \mathfrak{F}(\mathfrak{R})$. But N is a chief factor of $N.A_G(N)$ and we must then infer that $r(N) \in \mathfrak{R}_p$, against (1). Hence $N \nleqslant \Phi(G)$. In fact, we have shown that $C_G(N) \leqslant N$ and so G is primitive. We may assume that G is not cyclic of prime order p. For otherwise we have $1 \in f(p)$ implying that $\mathfrak{A}(\mathfrak{R}_p)$ and $\mathfrak{S}_{\pi_p}$ are both non-empty; then $1 \in \mathfrak{R}_p$ by R F 1, whence $G \in \mathfrak{F}(\mathfrak{R})$.

Let H be a complement of N in G. Since N may be thought of as a faithful, irreducible H-module over GF (p), we have $H \in f(p)$. We claim that

(2) $H \neq H^{\mathfrak{S}_{\pi_p}}$.

Suppose that $H = H^{\mathfrak{S}_{\pi_p}}$, so that $H \in \mathfrak{A}(\mathfrak{R}_p)$. Since H is then abelian, it must even be cyclic (by I.1.7) and $r(N) = \deg_{|H|} p = c \deg_d p$, where $c \mid |H|$ and d is a square-free integer with $d \mid |H|$ (we have used I.1.6 (c) here). Further, d can be chosen so that every prime dividing c divides d. Since H is an $\mathfrak{R}_{p*}$-group, d is a product of distinct primes from $\mathfrak{R}_{p*}$. Now, because $|N| \mid p^m - 1$ for some $m \in \mathfrak{R}_p$ ($H \in \mathfrak{A}(\mathfrak{R}_p)$!) it follows from

I.1.6 (b) that $c \deg_d p \mid m$. Since $H \in \dot{\mathfrak{F}}(\mathfrak{R})$, we have that c is $\mathfrak{R}$-potent and so $c \in \mathfrak{R}_p^*$. We have now shown that $\{c, d\}_p$ is an $\mathfrak{R}_p$-factorisation of $r(N)$, and from this we deduce via R F 2 that $r(N) \in \mathfrak{R}_p$, contradicting (1) and proving that (2) holds.

Set $A = H^{\mathfrak{S}_{\pi_p}}$. Then $A \in \mathfrak{A}(\mathfrak{R}_p)$. Let $e = \exp A$. By I.4.2 there are integers a and b such that:

(i) $e \mid b$;

(ii) $p \nmid b$;

(iii) $a \mid |H/A|$;

(iv) $ab \mid |H|$; and

(v) $r(N) = a \deg_b p$.

Since $a \mid |H/A|$ and $H/A \in \mathfrak{S}_{\pi_p}$, it follows from R F 1* that

(3) $a \in \mathfrak{R}_p^*$.

Let k be the product of the distinct primes dividing b and let l be that integer such that

$$\deg_b p = l.\deg_k p.$$

So if $q \in \mathbb{P}$ and $q \mid l$, then $q \mid k$. Let $q \mid k$. If $q \mid b/e$, then $q \in \pi_p \subseteq \mathfrak{R}_p^*$ since $H/A \in \mathfrak{S}_{\pi_p}$. On the other hand, suppose that $q \mid e$. Since $f(r)$ is S-closed for all primes r, we have that $\hat{\mathfrak{F}}(\mathfrak{R})$ is S-closed by II.3.23 (a). Therefore $N.A \in \hat{\mathfrak{F}}(\mathfrak{R})$. In fact, by (2) we have $|NA| < |G|$, and so $N.A \in \mathfrak{F}(\mathfrak{R})$. Considering $N|_A$ we see that a chief factor of $H.A$ below N has rank $d = \deg_e p \in \mathfrak{R}_p$. Now, $q \mid p^d - 1$, and so since then $\deg_q p \mid d$, we conclude from R F 1 that $\deg_q p \in \mathfrak{R}_p$, whence $q \in \mathfrak{R}_p^*$. Next suppose that $q \mid l$. If in fact $q \mid |H/A|$, then we have $q \in \mathfrak{R}_p^*$. So assume that $q \mid |A|$ but $q \nmid |H/A|$. Then we have $q \mid \deg_e p$. As above, we have $\deg_e p \in \mathfrak{R}_p$, and hence $q \in \mathfrak{R}_p^*$. Since $\mathfrak{R}$ has R F 1*, we deduce that $l \in \mathfrak{R}_p^*$.

and $\{l, k\}_p$ is an $\mathfrak{R}_p$-factorisation of $\deg_b p$. We now have

$$r(N) = a.l.\deg_k p.$$

Because $\mathfrak{R}$ has R F 1^* it follows that $a.l \in \mathfrak{R}_p^*$. Now let $s = \exp(\mathrm{Soc}(H))$. The analysis applied to A above shows that s is a product of distinct primes from $\mathfrak{R}_{p*}$. We set

$$n_b = [s, k],$$

a product of distinct primes from $\mathfrak{R}_{p*}$, and

$$n_a = r(N) / \deg_{n_b} p.$$

Clearly, $n_a \mid a.l \in \mathfrak{R}_p^*$ and so, because $\mathfrak{R}$ has R F 1^*, $n_a \in \mathfrak{R}_p^*$. Suppose that $q \mid n_a$ but $q \nmid n_b$. In particular, $q \nmid |\mathrm{Soc}(H)|$. As both a and l divide $|H|$, we have $q \mid |H|$. By II.1.17 there is a complemented t-chief factor X / Y for some prime t dividing $|\mathrm{Soc}(H)|$ such that $q \mid |A_H(X/Y)|$. Then

$$[X/Y] \cdot A_H(X/Y) \in Q(H) \subseteq \hat{\mathfrak{F}}(\mathfrak{R}) \cap \mathfrak{F}(\mathfrak{R}).$$

It is now elementary to show that there is a subgroup of $[X/Y] \cdot A_H(X/Y)$ isomorphic to $E(q, t)$. Since $\mathfrak{F}(\mathfrak{R})$ is S-closed and $|E(q, t)| \leq |H| < |G|$ we have

$$E(q, t) \in \mathfrak{F}(\mathfrak{R}).$$

By construction, $t \mid n_b$. We have now shown that $\{n_a, n_b\}_p$ is an $\mathfrak{R}_p$-factorisation of $r(N)$. But now, we have $r(N) \in A(\mathfrak{R}_p^*, \mathfrak{R}_{p*}) = \mathfrak{R}_p$ by R F 2^*. This contradiction proves that no such G exists, and

$$\hat{\mathfrak{F}}(\mathfrak{R}) \subseteq \mathfrak{F}(\mathfrak{R}),$$

as claimed.

Now suppose that $\hat{\mathfrak{F}}(\mathfrak{R}) \neq \mathfrak{F}(\mathfrak{R})$, and this time choose G to be of minimal order in $\mathfrak{F}(\mathfrak{R}) \setminus \hat{\mathfrak{F}}(\mathfrak{R})$. By II.4.13, as $\hat{\mathfrak{F}}(\mathfrak{R})$ is saturated, we have that G is primitive. Let N be the unique minimal normal subgroup of G and let H be a complement of N in G. Let p be the prime divisor of $|N|$. We must establish the fact that every normal subgroup of G is contained in $\mathfrak{F}(\mathfrak{R})$. Suppose, to the contrary, that this is false, and in particular that there is a maximal normal subgroup M of H such that $N.M \notin \mathfrak{F}(\mathfrak{R})$ (observe that we may assume that $H \neq 1$, for otherwise G is cyclic of prime order, p say. Since we would then have $f(p) \neq \emptyset$ we would have to conclude that $G \cong Z_p \in \hat{\mathfrak{F}}(\mathfrak{R})$, against the choice of G).

We have

$$H \in Q(G) \in \mathfrak{F}(\mathfrak{R})$$

and so, by choice of G, that $H \in \hat{\mathfrak{F}}(\mathfrak{R})$. Again we notice that $\hat{\mathfrak{F}}(\mathfrak{R})$ is S-closed, and consequently that $M \in \hat{\mathfrak{F}}(\mathfrak{R})$. Since we have already shown that $\hat{\mathfrak{F}}(\mathfrak{R}) \subseteq \mathfrak{F}(\mathfrak{R})$, we further obtain $M \in \mathfrak{F}(\mathfrak{R})$. If $N|_M$ is irreducible, then every chief factor of N.M is $\mathfrak{R}$-admissible, yielding $N.M \in \mathfrak{F}(\mathfrak{R})$. This contradiction forces $N|_M$ to be reducible, and the irreducible M-submodules of N to have dimension not belonging to $\mathfrak{R}_p$. Let

$$N|_M = N_1 \oplus \ldots \oplus N_r$$

be a decomposition of N into irreducible M-modules. Let $|H : M| = q$. Set $n = r(N)$, let $m = \dim N_1$ and let $\{m_a, m_b\}_p$ be a p-factorisation of m as guaranteed in I.4.4. That is, with $C = \mathrm{Ker}\,(M \text{ on } N_1)$, then

(vi) $\exp(\mathrm{Soc}\,(M / C) \mid m_b$;

(vii) $p \nmid m_b$;

(viii) m_b is square-free;

(ix) $m_a\, m_b \mid |M / C|$; and

(x) $m = m_a . \deg_{m_b} p$.

As $m_a\, m_b \mid |M / C|$ both m_a and m_b are $\mathfrak{R}$-potent. Because $\deg_{m_b} p \mid n = r(N)$ we obtain that m_b is a product of primes from $\mathfrak{R}_{p^*}$. Similarly $m_a \mid n$ so $m_a \in \mathfrak{R}_p^*$. Suppose that $q \mid m_a$ and $q \nmid m_b$. In particular, $q \nmid |\mathrm{Soc}(M / C)|$ and so there is complemented t-chief factor X / Y of M / C for some prime t dividing $|\mathrm{Soc}(M / C)|$ with $q \mid |A_{M/C}(X / Y)|$. Then $[X / Y] \cdot A_{M/C}(X / Y) \in Q(M / C) \subseteq Q(M) \subseteq \hat{\mathfrak{F}}(\mathfrak{R})$. It is easy to see that there is a subgroup of $[X / Y] \cdot A_{M/C}(X / Y)$ isomorphic to $E(q, t)$. Since $\hat{\mathfrak{F}}(\mathfrak{R})$ is subgroup closed we have

$$E(q, t) \in S\,Q(M) \subseteq \hat{\mathfrak{F}}(\mathfrak{R}) \subseteq \mathfrak{F}(\mathfrak{R}).$$

Hence $\deg_q t \in \mathfrak{R}_t$. Since $t \mid \exp(\mathrm{Soc}(M / C))$ and $\exp(\mathrm{Soc}(M / C)) \mid m_b$, we have shown that $\{m_a, m_b\}_p$ is an $\mathfrak{R}_p$-factorisation of $r(N_1)$. Hence

$$r(N_1) \in A(\mathfrak{R}_p^*, \mathfrak{R}_{p^*}) = \mathfrak{R}_p$$

by R F 2*. This contradiction proves the claim that $S_n(G) \subseteq \mathfrak{F}(\mathfrak{R})$.

Our choice of G is such that

$$\text{(4)} \quad H \notin \mathfrak{A}(\mathfrak{R}_p)\mathfrak{S}_{\pi_p}.$$

Obviously we may assume that H is not abelian. Let $K = H^{\mathfrak{S}_{\pi_p}}$, and suppose that $K < H$. Our choice of G forces $1 < K$ by (4). Since $N.K \triangleleft G$ we have $N.K \in \hat{\mathfrak{F}}(\mathfrak{R})$ and $K \in B_o f(p) = f(p)$

$= \mathfrak{A}(\mathfrak{R}_p)\mathfrak{S}_{\pi_p}$. Clearly $K = K^{\mathfrak{S}_{\pi_p}}$ and so $K \in \mathfrak{A}(\mathfrak{R}_p)$. But now we have $H \in \mathfrak{A}(\mathfrak{R}_p)\mathfrak{S}_{\pi_p}$, against (4). Therefore

$$\text{(5)} \quad H = H^{\mathfrak{S}_{\pi_p}}.$$

Since H is not abelian, and therefore not simple, we may choose a maximal normal subgroup M of H. By the argument used on K above, we have $M \in \mathfrak{A}(\mathfrak{R}_p)\mathfrak{S}_{\pi_p}$. Let $A = M^{\mathfrak{S}_{\pi_p}}$ and let $s = |H : M|$, a prime. Then A char $M \triangleleft H$ so $A \triangleleft H$ and $A \in \mathfrak{A}(\mathfrak{R}_p)$. If we are not to contradict (4), then it is evident that we must assume that $s \notin \pi_p$.

We denote by $\overline{B}$ the image $B.A / A$ of B under the natural homomorphism of H onto H / A. Let $S \in Syl_s(H)$ and $Q \in Hall_{\pi_p}(M) \subseteq Hall_{\pi_p}(H)$. We shall show that $[\overline{Q}, \overline{S}] = 1$; because H is $\mathfrak{S}_{\pi_p}$-perfect (by (5)) and $\overline{H} = \overline{Q}\,\overline{S}$, this shows that $Q \leq A$ and hence that H is a π_p'-group (A is of course a π_p'-group).

If then $(|H|, m) \neq 1$, where $m = \dim N$, then there is a prime dividing both $|H|$ and m. This prime would have to be $\mathfrak{R}$-potent because it divides $|H|$ and then will belong to $\mathfrak{R}_p^* \cap \mathbb{P} = \pi_p$ because it divides $m \in \mathfrak{R}_p$. This contradicts the fact that H is a π_p'-group. Therefore we shall have to have $(|H|, m) = 1$. But now [26],VI, 8.1 forces H to be cyclic, a contradiction.

So suppose that $\overline{Q} = 1$ and $[\overline{Q}, \overline{S}] \neq 1$. We observe that $s = |\overline{S}|$ and hence, because of (4), that $\overline{Q}$ is the unique maximal normal subgroup of $\overline{H}$. Since $\overline{Q} \leq \overline{H}$ and $\overline{Q} \cap \overline{S} = 1$, it is clear that if $\overline{Q} / \overline{Q}_1$ is a chief factor of $\overline{H}$ for some $Q_1 \triangleleft H$ then it is a complemented chief factor. Let q be the prime divisor of $|\overline{Q} / \overline{Q}_1|$. Since $[\overline{Q} / \overline{Q}_1] \cdot A_H(X / Y) \in Q(\overline{H})$, it follows from (4) that $A_H(\overline{Q} / \overline{Q}_1) \neq 1$, and hence

(6) $\overline{S} \cong \overline{S}\,\overline{Q} / \overline{Q} = A_{\overline{H}}(\overline{Q} / \overline{Q}_1) \cong Z_s$.

Let r be the rank of $\overline{Q} / \overline{Q}_1$ and suppose that $q \neq p$. Since $H \in \mathfrak{F}(\mathfrak{R})$ we have $r \in \mathfrak{R}_q$. Because $q \mid |\overline{Q}|$ and $\overline{Q} \in \mathfrak{S}_{\pi_p}$ we have

$$p \nmid q \in \pi_p = \mathcal{R}_p^* \cap \mathbb{P}.$$

Of course, $r = \deg_s q$ and so $s \in \mathcal{R}_{q*}$ (s is $\mathcal{R}$-potent since $s \mid |H|$). However, R F 5 now implies that $s \in \mathcal{R}_p^*$, contradicting our assumption that $s \in \pi'_p$.

The supposition that $q \nmid p$ must therefore have been false. Hence we have

(7) $p \in \pi_p$; and

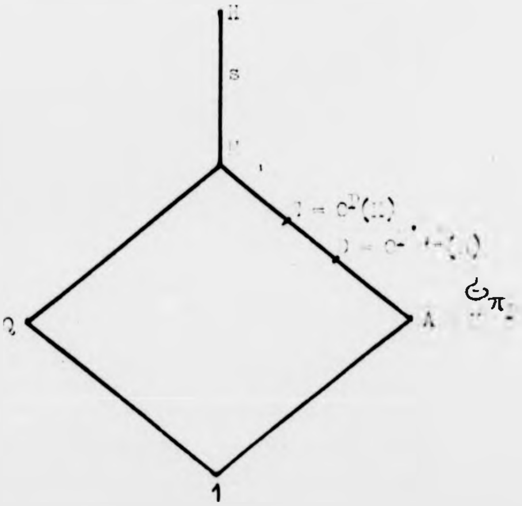
(8) $r = \deg_s p \in \mathcal{R}_p$.

From (8) and the fact that s is $\mathcal{R}$-potent, we obtain

(9) $s \in \mathcal{R}_{p*}$.

The argument used above shows that whenever X / Y is a π_p-chief factor of H such that $A_H(X / Y) \cong Z_s$, then $p \mid |X / Y|$.

<u>Case 1</u>: <u>Q is not a p-group</u>. We have shown above that $O^p(\bar{Q}) < \bar{Q}$, since $|\bar{Q} / \bar{Q}_1|$ is a power of p. If Q is not a



<u>Figure 3</u>

p-group then we must have that $O^{p',p}(\bar{Q}) = O^{p'}(O^p(\bar{Q})) < O^p(\bar{Q})$.

Let D and E be subgroups of H with $A \leq D < E \leq H$, $D = O^{p',p}(M)$ and $E = O^{p}(M)$. Clearly $\overline{D} = O^{p',p}(\overline{M})$ and $\overline{E} = O^{p}(\overline{M})$. Since D char M and E char M, both D and E are normal subgroups of H. If B is a subgroup of H, then we denote the image BD / D of B under the natural homomorphism of H onto H / D by B^{o}. Then $(|E^{o}|, |H^{o} : E^{o}|) = 1$ and so E^{o} is complemented in H^{o} and there is a complemented π_{p}-chief factor E^{o} / E_{1}^{o} of H^{o} with $E_{1} \triangleleft H$. Since H^{o} is $\mathfrak{S}_{\pi_{p}}$-perfect (by(5)), we see that $A_{H^{o}} (E^{o} / E_{1}^{o}) \neq 1$,

as $[E^{o} / E_{1}^{o}] \cdot A_{H^{o}} (E^{o} / E_{1}^{o}) \in Q (H^{o})$. In particular, $s \mid |H^{*}|$, where $H^{*} = A_{H^{o}} (E^{o} / E_{1}^{o})$. Were $H^{*} \cong Z_{s}$ then we have already seen that this would force $p \mid |E^{o} / E_{1}^{o}|$, contrary to the construction of E^{o} / E_{1}^{o}. Hence $p \mid |H^{*}|$ and $F(H^{*})$ is a p-group. Let N^{*} be a minimal normal subgroup of H^{*} and let r be the prime divisor of $|E^{o} / E_{1}^{o}|$. Since $H \in \hat{\mathfrak{Z}}(\mathcal{R})$ we have that H, and hence $[E^{o} / E_{1}^{o}] \cdot H^{*}$, lies in $\hat{\mathfrak{Z}}(\mathcal{R})$. The S-closure of $\hat{\mathfrak{Z}}(\mathcal{R})$ yields $[E^{o} / E_{1}^{o}] \cdot N^{*} \in \mathfrak{Z}(\mathcal{R})$. Applying Clifford's Theorem, we see that a chief factor of $[E^{o} / E_{1}^{o}] \cdot N^{*}$ below its socle has rank $\deg_{p} r \in \mathcal{R}_{r}$. Hence

(10) $p \in \mathcal{R}_{r^{*}}$

Also, $r \in \pi_{p}$ and so by R F 4, we have

(11) $r \in \mathcal{R}_{p^{*}}$

Next, since $(E^{o} / E_{1}^{o}).H^{*} \in \hat{\mathfrak{Z}}(\mathcal{R})$ we have $H^{*} \in F(r) = \mathfrak{A}(\mathcal{R}_{r})\mathfrak{S}_{\pi_{r}}$.

Because of (5) we must have $O_{s}(H^{*}) = 1$ ($s \mid |H^{*}|$). By II.1.12 there is a complemented p-chief factor X^{*} / Y^{*} of H^{*} on which a Sylow s-subgroup of H^{*} acts non-trivially. Since H^{*} is a $\{r, s\}$-group it is clear that

$$\left[X^{*}/Y^{*}\right] \cdot A_{H^{*}}(X^{*}/Y^{*}) \cong E\,(s\,.\,p) \in Q\,(H^{*}) \subseteq f\,(r).$$

Since $E\,(s\,,\,p)$ is faithfully and irreducibly represented on a module of dimension $[s\,,\,\deg_r p]$ over $GF\,(r)$, we deduce that

(12) $s \in \mathfrak{R}^{*}_{r}$.

Taking (7), (9), (10), (11) and (12) together, see that the conditions of R F 6 are satisfied with s in place of r and r in place of q. Since $\mathfrak{R}$ has R F 6, this forces $s \in \mathfrak{R}^{*}_{p}$, a contradiction. This case cannot, therefore occur.

Case 2 : Q is a p-group. Since H acts faithfully and irreducibly on N, we have $O_p\,(H) = 1$. Therefore $Q \cap C_H\,(A) = 1$; for $Q \cap C_H\,(A) \lneq A\,Q = M \lneq H$, and then $Q \cap C_H\,(A) \leq O_p\,(A\,Q) \leq O_p\,(H) = 1$. In particular, because $(\,|Q|\,,\,|A|\,) = 1$ there is a chief factor $J\,/\,L$ of H below A such that $Q \nleq C_H\,(J\,/\,L)$. Because A is abelian we have $A \leq C_H\,(J\,/\,L)$. Because $H = H^{\mathfrak{S}_{\pi_p}}$ (by (5)) we must have $s \,\big|\, |A_H\,(J\,/\,L)|$. Thus, with $H_1 = A_H\,(J\,/\,L)$, we see that $F\,(H_1)$ is a p-group and $|H_1 : F\,(H_1)| = s$. Arguing just as in case 1, we see that

(13) $p \in \mathfrak{R}_{r^{*}}$,

where r is the prime divisor of $|J\,/\,L|$. In this instance, r is a prime divisor of $|A|$, and $A \in \mathfrak{A}(\mathfrak{R}_p)$. Hence

(14) $r \in \mathfrak{R}_{p^{*}}$.

Lastly, $H \in \hat{\mathfrak{F}}(\mathfrak{R})$, so $H_1 \in f\,(r) = \mathfrak{A}(\mathfrak{R}_r)\mathfrak{S}_{\pi_r}$. Hence

(15) $s \in \pi_r$.

Taking (7), (9), (13), (14) and (15) together, and applying R F 6 (with r for q and s for r), we have $s \in \pi_p$.

This contradiction has finally proved our claim that $[\bar{T}\,,\,\bar{S}] = 1$, for $(\,|\bar{T}|\,,\,|\bar{S}|\,) = 1$ and $\bar{S}$ must centralise every π_p-chief factor of $\overline{H}$. This means that $Q \leq N_H(S.A)$ - i.e. that $A.S \trianglelefteq H$. The fact that H must be $\mathfrak{S}_{\pi_p}$-perfect (by (5)) now forces $A.S = H$, and the result follows as already described.

q.e.d.

<u>III.2.4 Corollary</u> Let $\mathfrak{s}$ be a ranking function. If $\mathfrak{F}(\mathfrak{s})$ is saturated, then $\mathfrak{F}(\mathfrak{s})$ is also S-closed.

<u>Proof</u> The local definition given in III.2.3 for $\mathfrak{F}(\mathfrak{s})$ is S-closed. Now apply II.3.23 (a).

q.e.d.

<u>III.2.5 Corollary</u> Let $\mathfrak{R}$ be a full ranking function. The following statements are equivalent.

(<u>a</u>) $\mathfrak{F}(\mathfrak{R})$ is saturated.

(<u>b</u>) $\mathfrak{R}$ has the following properties:

F R F 1: For all primes p, $\mathfrak{R}_p$ is closed under division and multiplication.

F R F 2: If p and q are primes such that $p \nmid q \in \mathfrak{R}_p$ and if $n \in \mathfrak{R}_q$, then $q^n - 1 \in \mathfrak{R}_p$.

F R F 3: If p, q and r are primes such that

(i) $r \mid p^a - 1$ for some $a \in \mathfrak{R}_p$;

(ii) $p \mid q^b - 1$ for some $b \in \mathfrak{R}_q$;

(iii) $q \mid p^c - 1$ for some $c \in \mathfrak{R}_p$;

(iv) $r \in \mathfrak{R}_q$, and

(v) $p \in \mathfrak{R}_p$,

then $r \in \mathfrak{R}_p$.

(<u>c</u>) $\mathfrak{F}(\mathfrak{R})$ is locally defined by

$$f(p) = \mathfrak{O}(\mathfrak{R}_p)\mathfrak{S}_{\pi_p},$$

where we set $\pi_p = \mathfrak{R}_p \cap \mathbb{P}$ for all $p \in \mathbb{P}$.

<u>Proof</u> <u>(a)$\Rightarrow$(b)</u> By III.1.16, $\mathfrak{R}_p = \mathfrak{R}_p^*$. By III.1.17 we have that $\mathfrak{R}$ has F R F 1. Let p and q be primes such that $p \neq q \in \mathfrak{R}_p$ and let s be a prime with $s \mid q^n - 1$ for some $n \in \mathfrak{R}_q$. By III.2.3, $\mathfrak{R}$ has R F 1, and so, since $\deg_s q \mid n \in \mathfrak{R}_q$, we have $s \in \mathfrak{R}_{q*}$. Since, again by III.2.3, $\mathfrak{R}$ has R F 5, it now follows that $s \in \mathfrak{R}_p^* = \mathfrak{R}_p$. Hence every prime dividing $q^n - 1$ belongs to $\mathfrak{R}_p$, and since $\mathfrak{R}$ has F R F 1, even $q^n - 1 \in \mathfrak{R}_p$. Thus $\mathfrak{R}$ has F R F 2. A similar argument applied to the hypothesis of F R F 3 shows that $r \in \mathfrak{R}_{p*}$, $p \in \mathfrak{R}_{q*}$ and $q \in \mathfrak{R}_{p*}$. We may now apply R F 6 immediately to get $r \in \mathfrak{R}_p$.

<u>(b)$\Rightarrow$(c)</u> We need to check that $\mathfrak{R}$ has R F 1^*, R F 2^*, R F 4, R F 5 and R F 6. This part will then follow from III.2.3. Evidently, F R F 2 and F R F 3 imply R F 5 and R F 6 respectively. Observe that $\mathfrak{R}_p = \mathfrak{R}_p^*$; for certainly $\mathfrak{R}_p \subseteq \mathfrak{R}_p^*$, and since $\mathfrak{R}_p$ is closed under taking divisors, it follows that $\mathfrak{R}_p = \mathfrak{R}_p^*$. Hence R F 1^* holds. We now show that $\mathfrak{R}$ has R F 2^*. We note that F R F 1 clearly gives $A(\mathfrak{R}_p^*, \mathfrak{R}_{p*}) \subseteq \mathfrak{R}_p$. Let $m \in \mathfrak{R}_p$ and set $G \cong Z_{p^m - 1}$. Since $\mathfrak{R}$ is full and $\mathfrak{R}$ has F R F 1, the group G belongs to $\mathfrak{F}(\mathfrak{R})$. Also, G is faithfully and irreducibly represented on a module V of dimension m over GF (p). Hence $[V]\cdot G \in \mathfrak{F}(\mathfrak{R})$. Now, $m = \deg_{|G|} p$. Let $q_1, \ldots, q_t$ be the distinct primes dividing $|G|$. Applying I.1.6 (c), there are non-negative integers $e_1, \ldots, e_t$ such that

$$m = q_1^{e_1} \ldots q_t^{e_t} \cdot \deg_{q_1 \ldots q_t} p.$$

Since $\deg_{q_i} p \mid m \in \mathfrak{R}_p$ and $\mathfrak{R}$ has F R F 1, it follows that $q_i \in \mathfrak{R}_{p*}$ for $1 \leq i \leq t$. Also, if $e_i > 0$ then $q_i \mid (m, |G|)$, so $q_i \in \mathfrak{R}_p^*$. Hence $m \in A(\mathfrak{R}_p^*, \mathfrak{R}_{p*})$. Consequently $\mathfrak{R}$ has R F 2^*. It remains to check that $\mathfrak{R}$ has R F 4. If $p \neq q \in \mathfrak{R}_p$, then by F R F 2 we have $q - 1 \in \mathfrak{R}_p$. Since $\deg_q p \mid q - 1$ and because $\mathfrak{R}$ has F R F 1, we conclude that $\deg_q p \in \mathfrak{R}_p$.

Hence $q \in \mathfrak{R}_{p*}$.

(c)$\Rightarrow$(a) This is proven in III.2.3.

q.e.d.

The implication (b)$\Rightarrow$(c) of III.2.5 was originally proved by Heineken [25] using very different methods from those used here.

We now recall the arithmetically defined classes defined in II.4.11. In II.4.14 it was proved that all Gaschütz classes which are formations are precisely the ranked saturated formations. The characterisation of III.2.3 therefore determines the sets Ω which give all Gaschütz classes $\mathfrak{R}_\Omega$ which are formations. We can improve the characterisation of these Gaschütz classes.

III.2.6 Theorem The following statements are equivalent in pairs for a class $\mathfrak{H}$:

(a) $\mathfrak{H}$ is a Gaschütz class and a formation.

(b) $\mathfrak{H}$ is a ranked saturated formation.

(c) $\mathfrak{H}$ is a subgroup-closed Gaschütz class.

Proof (a)$\Leftrightarrow$(b) II.4.14.

(b)$\Rightarrow$(c) $\mathfrak{H}$ is a Gaschütz class by (b)$\Rightarrow$(a), and is subgroup-closed by III.2.4.

(c)$\Rightarrow$(a) By II.3.5, $\mathfrak{H}$ is D_0-closed. Hence, by II.2.7 (b) we have

$$R_0\mathfrak{H} \subseteq S D_0\mathfrak{H} = \mathfrak{H}$$

q.e.d.

3. Examples and conjectures

We begin by justifying that the formations introduced in III.1.10 are saturated.

III.3.1 Examples (a) Let $\mathfrak{Q}$ be a ranking function defined by

$$\mathfrak{Q}_2 = \mathfrak{Q}_3 = \{1\};$$

$$\mathfrak{Q}_5 = \{2^i . 3^j : i, j \in \mathbb{N} \cup \{0\}\};$$

$$\mathfrak{Q}_p = \{1\} \text{ for } p \geqslant 7.$$

It is clear that $\mathfrak{Q}$ has F R F 1. Since $\mathfrak{Q}_2 = \mathfrak{Q}_3 = \{1\}$, it is easily checked that $\mathfrak{Q}$ has F R F 2. Finally, $p \notin \mathfrak{Q}_p$ for all primes p, and so (v) of F R F 3 is never satisfied. $\mathfrak{Q}$ has F R F 3 vacuously. It is now a consequence of III.2.5, (b)$\Longrightarrow$(a) that $\mathfrak{F}(\mathfrak{Q})$ is a saturated formation.

(b) Let $\mathfrak{U}$ be a ranking function defined by

$$\mathfrak{U}_2 = \{3 . 2^i : i \geqslant 0\};$$

$$\mathfrak{U}_3 = \emptyset;$$

$$\mathfrak{U}_p = \{1\} \quad \text{for } p \geqslant 5.$$

We show that $\mathfrak{U}$ has R F 1*, R F 2*, R F 4, R F 5 and R F 6. Certainly R F 1* holds for $p > 2$. Clearly $\mathfrak{U}_2^* = \{2^i : i \geqslant 0\}$ and so $\mathfrak{U}$ has R F 1*. Now, $3 = \deg_7 2$ and so if $m \in \mathfrak{U}_2$ then $m = 2^i \deg_7 2 \in A(\mathfrak{U}_2^*, \mathfrak{U}_{2*})$. On the other hand, if k is a product of primes from $\mathfrak{U}_{2*}$ then $\deg_k p = 2^j . 3$ and so any element of $A(\mathfrak{U}_2^*, \mathfrak{U}_{2*})$ has the form $1 . \deg_k p = 2^i . 2^j . 3 = 2^{i+j} . 3 \in \mathfrak{U}$ Consequently, $\mathfrak{U}$ has R F 2*. Since $2 = \mathfrak{U}_2^* \cap \mathbb{P}$ and $\mathfrak{U}_p \cap \mathbb{P} = \emptyset$ for $p \geqslant 3$, $\mathfrak{U}$ has R F 4 vacuously. No element of $\mathfrak{U}_p$ contains a $\mathfrak{U}$-potent prime different from p, for all primes p, and hence R F 5 holds vacuously. Lastly, $p \in \mathfrak{U}_p$ if and only if $p = 2$. Therefore, R F 6 could only be applied if there is a prime $r \neq 2$ such that $r \in \mathfrak{U}_q$ for some prime $q \neq p$. This clearly never happens. Thus $\mathfrak{U}$ satisfies (b) of III.2.3 and so $\mathfrak{F}(\mathfrak{U})$ is saturated.

III.3.2 Example Recall example (✱) which was introduced in Section 2 of this chapter. In its final form, $\mathfrak{U}$ was a ranked function with

$$\mathfrak{U}_2 = \mathfrak{U}_3 = \{2^i.3^j : i , j \geqslant 0\};$$
$$\mathfrak{U}_{11} = \{1\} ; \text{ and}$$
$$\mathfrak{U}_p = \emptyset \text{ for all } p \in \mathbb{P} \setminus \{2 , 3 , 11\}.$$

This ranking function was constructed to have properties R F 1, ... , R F 6, and so by III.2.3, $\mathfrak{F}(\mathfrak{U})$ is saturated.

Unfortunately, the arithmetical properties required of $\mathfrak{R}$ for $\mathfrak{R}$ to satisfy (b) and (c) of III.2.3 are sufficiently complicated as to deny a satisfactory description of $\mathfrak{F}(\mathfrak{R})$. In the full characteristic case, however, we have already noted that Heineken has proved some further properties which might help to improve our insight. For instance,

III.3.3 Lemma (Heineken [25], Corollary 4). Let $\mathfrak{R}$ be a full ranking function having F R F 1, F R F 2 and F R F 3. Suppose that p and q are distinct primes such that $p \mid q^n - 1$ for some $n \in \mathfrak{R}_q$ and $q \mid p^m - 1$ for some $m \in \mathfrak{R}_p$.

(a) If $p \in \mathfrak{R}_p$ and $q \in \mathfrak{R}_q$, then $\mathfrak{R}_p = \mathfrak{R}_q = \mathbb{N}$.

(b) If $\mathfrak{R}_p = \{p^i : i \geq 0\}$, then $\mathfrak{R}_q = \{1\}$.

Proof (a) Set q = r in the statement of F R F 3. Then $q \in \mathfrak{R}_p$. Reversing the roles of p and q in F R F 3 gives $p \in \mathfrak{R}_q$. Applying III.1.25 yields $\mathfrak{R}_p = \mathfrak{R}_q = \mathbb{N}$.

(b) See [25].

q.e.d.

A few minutes thought reveals that even with III.1.25 and III.3.3, the ranked saturated formations of full characteristic are still too complicated to be easily calculated. However, consideration of the primes involved lead one to make the following conjecture.

III.3.4 Conjecture Let q and r be primes. We define sets A_i recursively as follows:

$$A_0 = \{ p \in \mathbb{P} : p \mid q^{r^j} - 1,\ j \geq 1 \};$$

$$A_i = \{ p \in \mathbb{P} : p \mid t - 1 \text{ for } t \in A_{i-1} \},\ i \geq 1.$$

Then $\mathbb{P} = \bigcup_i A_i$.

If III.3.4 is indeed true, then the author proposes a further conjecture.

III.3.5 Conjecture Let p, q and r be primes with $p \neq q$. Let $\mathcal{R}$ be a full ranking function such that $\mathfrak{J}(\mathcal{R})$ is saturated, and suppose that

$$r \in \mathcal{R}_q \text{ and } q \in \mathcal{R}_p.$$

Define $B_0 = \{ s \in \mathbb{P} : s \mid q^{r^j} - 1,\ j \geq 1 \}$

$$B_i = \{ s \in \mathbb{P} : s \mid t - 1 \text{ for } t \in B_{i-1} \setminus \{p\} \},\ i \geq 1.$$

Since $\mathcal{R}$ has $\partial \mathcal{R} \geq 2$ it follows that $\bigcup_{i \geq 0} B_i \subseteq \mathcal{R}_p$. The conjecture is then that $\mathbb{P} = \bigcup_{i \geq 0} B_i$ and $\mathcal{R}_p = \mathbb{N}$.

If III.3.4 holds then certainly $\mathcal{R}_p$ will contain all primes greater than or equal to p. Also, if III.3.5 is indeed true and $\mathcal{R}$ is as stated, then whenever $p \in \mathbb{P}$ such that $\{1\} \neq \mathcal{R}_p \neq \mathbb{N}$ then every prime t with $p \,/\, t \in \mathcal{R}_p$ must satisfy $\mathcal{R}_t = \{1\}$. How may such $\mathcal{R}$ be constructed?

III.3.6 Constructing a full ranking function $\mathcal{R}$ such that $\mathfrak{J}(\mathcal{R})$ is saturated, assuming III.3.5. We decompose $\mathbb{P}$ into a union of subsets

$\Omega_1 \cup \Omega_1^* \cup \Omega_2 \cup \Omega_2^* \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$ - defined as follows:

Ω_1 is the set of primes p such that $\{1\} \neq \mathcal{R}_p \neq \mathbb{N}$ and $p \notin \mathcal{R}_p$.

By III.3.5, we require that if $\pi_p = \mathcal{R}_p \cap \mathbb{P}$ and

$\pi_p^* = \{q \in \mathbb{P} : q \mid t - 1$ some $t \in \pi_p\}$, then

$$p \notin \pi_p = \pi_p^* .$$

Let $\Omega_1^* = \{t \in \mathbb{P} : t \in \pi_p$ for some prime $p \in \Omega_1\}$. Our remark following the conjecture III.3.5 implies that

if $p \in \Omega_1^*$, then $\mathcal{R}_p = \{1\}$.

Define Ω_2 to be the set of primes p such that $\{1\} \neq \mathcal{R}_p \neq \mathbb{N}$ and $p \in \mathcal{R}_p \neq \{p^i : i \geq 0\}$. With π_p and π_p^* as above we require that

$$\pi_p \setminus \{p\} = \pi_p^*, \text{ and } p \in \pi_p .$$

Additionally, if $q \in \Omega_1$ and $r \in \pi_q$ are such that $q \mid p^m - 1$ and $r \mid p^n - 1$ for some m, $n \in \mathcal{R}_p$, then $r \in \mathcal{R}_p$.

If $\Omega_2^* = \{t \in \mathbb{P} : t \in \pi_p^*$ for some $p \in \Omega_2\}$, then

if $p \in \Omega_2^*$, then $\mathcal{R}_p = \{1\}$.

This is consistent with the definition of Ω_1^*; for if r and q exist as above then $r \neq q$ since $q \notin \pi_q$. Hence $r \in \Omega_1^*$. Further, if $\{p, q\} \subseteq \Omega_2$ with $p \neq q$ then either $p \nmid q^m - 1$ for any $m \in \mathcal{R}_q$ or $q \nmid p^n - 1$ for any $n \in \mathcal{R}_p$. For suppose that $q \mid p^n - 1$ and $p \mid q^m - 1$ for some $n \in \mathcal{R}_p$ and $m \in \mathcal{R}_q$. Then III.3.3 (a) would force $\mathcal{R}_p = \mathcal{R}_q = \mathbb{N}$, against the choice of p and q.

If $p \in \Omega_3$ then $\mathcal{R}_p = \{p^i : i \geq 0\}$. The elements of Ω_3 must be chosen so that if $q \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ and $p \in \Omega_3$ then either $p \nmid q^m - 1$ for any $m \in \mathcal{R}_q$ or $q \nmid p^n - 1$ for any $n \in \mathcal{R}_p$. If the converse were true, then applying III.3.3 (b) would contradict the definition of $\mathcal{R}_q$.

Lastly we partition $\mathbb{P}\setminus(\Omega_1\cup\Omega_1^*\cup\Omega_2\cup\Omega_2^*\cup\Omega_3)$ into two sets Ω_4 and Ω_5.

<u>If $p\in\Omega_4$ then $\mathcal{R}_p=\mathbb{N}$; if $p\in\Omega_5$ then $\mathcal{R}_p=\{1\}$</u>.

If $q\in\Omega_4\cup\Omega_5$ and $q \mid p^m-1$ for some $p\in\Omega_2\cup\Omega_3$ and $m\in\mathcal{R}_p$ then q must belong to Ω_5; this again follows from III.3.3 (a).

<u>III.3.7 Lemma</u> Let $\Omega_1,\Omega_1^*,\Omega_2,\Omega_2^*,\Omega_3,\Omega_4,\Omega_5$ be as above and let $\mathcal{R}$ be the associated ranking function (If $p\in\Omega_1\cup\Omega_2$ then $\mathcal{R}_p$ is the set of all products of primes from π_p). Then $\mathcal{F}(\mathcal{R})$ is saturated.

<u>Proof</u> By construction, $\mathcal{R}$ has FRF 1 and FRF 2. We must therefore check FRF 3. Suppose that p, q and r are primes such that

(i) $r \parallel p^a-1$ some $a\in\mathcal{R}_p$;

(ii) $p \parallel q^b-1$ some $b\in\mathcal{R}_q$;

(iii) $q \parallel r^c-1$ some $c\in\mathcal{R}_r$;

(iv) $p\in\mathcal{R}_r$;

(v) $r\in\mathcal{R}_q$.

First notice that (iv) forces $r\in\Omega_2\cup\Omega_3\cup\Omega_4$. If $r\in\Omega_4$, then certainly $r\in\mathcal{R}_p=\mathbb{N}$. So we may assume that $r\in\Omega_2\cup\Omega_3$. If $r\in\Omega_3$, then q cannot belong to any of $\Omega_1,\Omega_2,\Omega_3$ or Ω_4 since, by construction, one of (ii) and (iii) must fail. If $q\in\Omega_1^*\cup\Omega_2^*\cup\Omega_5$, then $\mathcal{R}_q=\{1\}$ and (v) cannot then hold. Hence $r\in\Omega_2$. Again by construction, q cannot belong to any of Ω_2,Ω_3 or Ω_4, since otherwise one of (ii) and (iii) must fail, and if $q\in\Omega_1^*\cup\Omega_2^*\cup\Omega_5$, then (v) fails. Suppose that $q\in\Omega_1$. Then by construction $r\in\mathcal{R}_p$. Hence FRF 3 holds.

q.e.d.

In fact we have proven

<u>III.3.8 Theorem</u> If conjecture III.3.5 holds, then every ranked saturated formation of full characteristic is constructed as in III.3.6.

<u>III.3.9 Lemma</u> Let $\mathcal{R}$ be a full ranking function such that $\mathcal{R}_2 \neq \{1\}$ and $\mathfrak{F}(\mathcal{R})$ is saturated. If III.3.5 holds then $\{2\} \supseteq \Omega_1 \cup \Omega_2$.

<u>Proof</u> Let $p \in \mathbb{P} \setminus \{2\}$ such that $\{1\} \neq \mathcal{R}_p \neq \{p^i : i \geq 0\}$. By III.1.23 we have $2 \in \mathcal{R}_p^* = \mathcal{R}_p$. Since $\mathcal{R}_2 \neq \{1\}$ it follows from III.3.4 that $\mathcal{R}_p = \mathbb{N}$.

<u>III.3.10 Examples</u> (a) Let $\Omega_1 = \emptyset$, and let $\Omega_2 = \{2\}$. Set $\mathcal{R}_2 = \{2^i 5^j : i, j \geq 0\}$. Put $\Omega_3 = \{7, 11\}$ and $\Omega_5 = \mathbb{P} \setminus \{2, 7, 11, 5\}$. We must have $\pi_p^* = \{5\}$. Hence

$\mathcal{R}_2 = \{2^i 5^j : i, j \geq 0\}$;

$\mathcal{R}_3 = \{1\} = \mathcal{R}_5$;

$\mathcal{R}_7 = \{7^i : i \geq 0\}$;

$\mathcal{R}_{11} = \{11^j : j \geq 0\}$;

$\mathcal{R}_p = \{1\}$ for $p \geq 3$.

$\mathfrak{F}\mathcal{R}$ is a saturated formation by III.3.7.

(b) Let $\Omega_1 = \{3\}$ with $\mathcal{R}_3 = \{2^i 5^j : i, j \geq 0\}$. Then $\Omega_1^* = \{2, 5\}$. Let $\Omega_2 = \{11\}$ with $\mathcal{R}_{11} = \{2^i . 5^j . 11^k : i, j, k \geq 0\}$. Then $\Omega_2^* = \Omega_1^*$. Let $\Omega_3 = \{13\}$, and let $\Omega_4 = \{p : p \mid q^m - 1$ some $q \in \Omega_1 \cup \Omega_3$ and $m \in \mathcal{R}_q\}$ and $\Omega_5 = \mathbb{P} \setminus (\Omega_1 \cup \Omega_1^* \cup \Omega_2 \cup \Omega_2^* \cup \Omega_3 \cup \Omega_4)$.

Hence

$\mathcal{R}_2 = \{1\}$;

$\mathcal{R}_3 = \{2^i 5^j : i, j \geq 0\}$;

$\mathcal{R}_5 = \{1\}$;

$$\mathfrak{R}_7 = \mathbb{N} \; (\deg_7 3 = 6 \notin \mathfrak{R}_3,\ \deg_7 11 = 3 \notin \mathfrak{R}_{11},\ \deg_7 13 = 2 \notin \mathfrak{R}_{13});$$

$$\mathfrak{R}_{11} = \{ 2^i . 5^j . 11^k : i , j , k \geq 0 \};$$

$$\mathfrak{R}_{13} = \{ 13^i : i \geq 0 \};$$

$$\mathfrak{R}_{17} = \{1\} (17 \mid 3^{16} - 1 \text{ and } 16 \in \mathfrak{R}_3);$$

etc.

By III.3.7, $\mathfrak{F}(\mathfrak{R})$ is a saturated formation.

As we have already seen, an arbitrary ranked formation is considerably more complex. We can do little more than make two more conjectures dealing with some special cases.

<u>III.3.11 Conjecture</u> Let σ be a finite set of primes and let p and r be primes in $\mathbb{P} \setminus \sigma$. Define sets A_i recursively as follows:

$$A_0 = \{ q \in \mathbb{P} : \quad q \mid p^{r^j} - 1 \text{ some } j \geq 0 \}$$

$$A_i = \{ q \in \mathbb{P} : q \mid t - 1 \text{ some } t \in \mathbb{P} \setminus (\sigma \cup \{r\}) \}.$$

Then $\bigcup_{i \geq 0} A_i = \mathbb{P} \setminus \sigma$.

<u>III.3.12 Lemma</u> Let $\mathfrak{R}$ be a ranking function such that $\mathfrak{F}(\mathfrak{R})$ is saturated. Let $\sigma = \{ p : \mathfrak{R}_p = \emptyset \}$ and suppose that $|\sigma| < \infty$. If $p \neq q \in \mathfrak{R}_p^*$, q a prime, and $\emptyset \neq \mathfrak{R}_q^* \neq \{1\}$, and if III.3.11 holds then

$$\mathfrak{R}_p^* \cap \mathbb{P} = \mathbb{P} \setminus \sigma.$$

<u>Proof</u> Let r be a prime in $\mathfrak{R}_q^*$. If s is an $\mathfrak{R}$-potent prime dividing $q^{r^j} - 1$ for some $j \geq 0$, then there is an element m of $\mathfrak{R}_q$ with $r^j \mid m$, so that $s \mid q^m - 1$. Since $\mathfrak{R}$ has $R \neq 2$ it

follows from $\deg_s q \mid m$ that $s \in \mathcal{R}_{q*}$. By [illegible] 5, then, $s \in \mathcal{R}_p^*$.

Defining $B_0 = \left\{ t \in \mathbb{P} : t \mid q^{r^j} - 1 \text{ some } j \geqslant 0 \right\}$

$$B_i = \left\{ t \in \mathbb{P} : t \mid u - 1 \quad \text{some } u \in \mathbb{P} \setminus (\sigma \cup \{p\}) \right\}, \quad i \geqslant 1,$$

it therefore follows that $\bigcup_{i \geqslant 0} B_i \subseteq \mathcal{R}_p^*$. If III.3.11 holds, then we have

$$\mathcal{R}_p^* \cap \mathbb{P} = \mathbb{P} \setminus \sigma.$$

q.e.d.

<u>III.3.13 Conjecture</u> Let $\mathcal{R}$ be a ranking function such that $\mathfrak{F}(\mathcal{R})$ is saturated. Let $\sigma = \{ p : \mathcal{R}_p = \emptyset \}$ and suppose that $|\sigma| < \infty$. Then there is a full ranking function $\mathcal{S}$ such that $\mathfrak{F}(\mathcal{S})$ is saturated and

$$\mathfrak{F}(\mathcal{R}) = \mathfrak{F}(\mathcal{S}) \cap \mathfrak{S}_{\sigma'}.$$

Certainly, if III.3.11 holds then III.3.12 is an analogy of III.1.25 and therefore implies that a ranked saturated formation $\mathfrak{F}(\mathcal{R})$, where $\sigma = \{ p : \mathcal{R}_p = \emptyset \}$ is finite, is constructed in a similar way to that described in III.3.6. If σ is not finite, then we have already seen in Example III.1.26 that III.3.12 is false. We finish with one more example.

<u>III.3.14 Example</u> (Schacher and Seitz [28]) In [28], Schacher and Seitz show that there is an infinite set Ω of primes such that if p and q belong to Ω, then $2 \nmid \deg_q p$ (if $p \neq q$). We define $\mathcal{R}$ such that

(a) if $p \in \mathbb{P} \setminus \Omega$ then $\mathcal{R}_p = \emptyset$;

(b) if $p \in \Omega$ then $\mathcal{R}_p^* \cap \mathbb{P} = \Omega$ and $\mathcal{R}_{p*} = \Omega \setminus \{p\}$. Let $\mathcal{R}_p^*$ be the set of all products of primes in Ω.

(c) $\mathfrak{R}_p = A(\mathfrak{R}_p^*, \mathfrak{R}_{p^*})$ for all p .

Since $\Omega \subseteq \mathfrak{R}_p^*$ for all $p \in \Omega$ it is clear that $\mathfrak{R}$ has R F 5 and R F 6. Since $\mathfrak{R}$ has R F 1^*, R F 2^* and R F 4 by construction, then $\mathfrak{F}(\mathfrak{R})$ is saturated. Thus the class of all groups of odd order in which each chief factor has odd rank is a saturated formation.

Chapter IV

Chapter IV. Absolutely ranked saturated formations

Recall from Definition II.4.1 (c) and II.4.6 (b) that a formation $\mathfrak{F}$ is absolutely ranked if there is a ranking function $\mathfrak{R}$ such that $\mathfrak{F} = \mathfrak{F}_a(\mathfrak{R})$, where $\mathfrak{F}_a(\mathfrak{R}) = (G \in \mathfrak{G}$: for all $p \in \mathbb{P}$ and all p-chief factors H / N of G, $r_a(H / N) \in \mathfrak{R}_p)$. It was shown in II.4.8 that $\mathfrak{F}_a(\mathfrak{R})$ is a formation whenever $\mathfrak{R}$ is a ranking function. In this chapter we characterise the ranking functions $\mathfrak{R}$ such that $\mathfrak{F}_a(\mathfrak{R})$ is saturated. We shall see that the resulting absolutely ranked saturated formations are easily described, in contrast to the ranked saturated formations investigated in Chapter III. Until further notice, $\mathfrak{R}$ will be a ranking function such that $\mathfrak{F}_a(\mathfrak{R})$ is saturated, and $\mathfrak{S}$ will be an arbitrary ranking function. The reader is reminded that all ranking functions are assumed to be minimal or absolute minimal (depending on the context) and all groups are finite and soluble. We begin with an elementary observation derived from Chapter I.

IV.1 Lemma Let K and L be fields with $K \subseteq L$ and $|L : K| < \infty$. Let G be a group and suppose that L is a splitting field for G. Then there is an irreducible K G-module V of absolute dimension n (i. e. $r_a(V) = n$) if and only if there is an irreducible L G-module W of dimension n. If χ is the character afforded by W, and L is a field of characteristic p, then

$$\dim_K V = |K(\chi) : K| . \dim_L W.$$

Proof I.2.16 (a).
q.e.d.

IV.2 Lemma Let p be a prime. Then

$$\mathfrak{R}_p \neq \emptyset \Leftrightarrow 1 \in \mathfrak{R}_p .$$

Proof Precisely as in III.1.1.

q.e.d.

IV.3 Lemma Let p and q be $\mathfrak{R}$-potent primes.

(a) Let $t \in \mathbb{N}$ be $\mathfrak{R}$-potent. Then $Z_t \in f(p)$.

(b) If $p \neq q \in \mathfrak{R}_p$, then $f(p)$ contains an extraspecial group of order q^3, and hence contains every extraspecial q-group.

(c) If $p \in \mathfrak{R}_p$ and $q \neq p$, then $E(p, q) \in f(p)$.

(d) If $a \in \mathfrak{R}_q$, then a is $\mathfrak{R}$-potent.

Proof (a) Let $G \cong Z_t$ and $\overline{G} = G / O_p(G)$. As t is $\mathfrak{R}$-potent it follows from IV.2 that $G \in \mathfrak{J}_a(\mathfrak{R})$. Then $\overline{G}$ is faithfully and irreducibly represented on a module W of dimension 1 over a splitting field L for G of characteristic p. Let V be an irreducible K G-submodule of W_K with $K = GF(p)$. Since $\mathfrak{R}_p \neq \emptyset$ we have $1 \in \mathfrak{R}_p$ by IV.2 and hence $[V] \cdot \overline{G} \in \mathfrak{J}_a(\mathfrak{R})$ and $\overline{G} \in f(p)$. As f is full, we then deduce that $G \in f(p)$.

(b) Let E be an extraspecial group of order q^3. Since each chief factor of E has rank 1 and $1 \in \mathfrak{R}_q$ we have $E \in \mathfrak{J}_a(\mathfrak{R})$. Let L be a splitting field for E of characteristic p and let $K = GF(p) \subseteq L$. If W is a faithful irreducible E-module over L then $\dim_L W = q$ by I.4.7, I.4.8 and I. 2.16. Thus if V is an irreducible K E-submodule of W_K, then $[V] \cdot E \in \mathfrak{J}_a(\mathfrak{R})$ and $E \in f(p)$. Since E is an arbitrary extraspecial group of order q^3, it follows from [26] III, 13.7 and III.13.8 that every extraspecial q-group belongs to $f(p)$.

(c) Let $G = E(p, q)$ be the group in the statement. It follows from (a) that $G \in \mathfrak{J}_a(\mathfrak{R})$. By I.4.10, I.4.11 and I.2.16, the group G has a faithful and irreducible module W of dimension p over a splitting field L of characteristic p.. Since $p \in \mathfrak{R}_p$, we deduce that if V is an irreducible K G-submodule of W_K, where $K = GF(p)$, then $[V] \cdot G \in \mathfrak{F}_a(\mathfrak{R})$. In particular, $G \in f(p)$.

(d) The minimality of $\mathfrak{R}$ means that there is a group $G \in f(p)$ with a faithful and irreducible module V over $GF(p)$ such that $r_a(V) = a$. By I.4.3, we have $a \mid |G|$. Since $G \in \mathfrak{F}_a(\mathfrak{R})$, it follows that a is $\mathfrak{R}$-potent.

q.e.d.

The next result compares with condition R f 1* in Chapter III:

IV.4 Proposition (a) If $a \in \mathfrak{R}_p$ and $b \mid a$, then $b \in \mathfrak{R}_p$.

(b) If $\{a, b\} \subseteq \mathfrak{R}_p$, then $a.b \in \mathfrak{R}_p$.

Proof (a) We assume, without loss of generality, that $a \neq 1$.

Let $q_1^{c_1} \dots q_k^{c_k} p^c$ be the prime decomposition of a, where $\{p, q_1, \dots, q_k\}$ is a set of distinct primes, $c_i \in \mathbb{N}$ for $1 \leqslant i \leqslant k$ and $c \in \mathbb{N} \cup \{0\}$.

Then $b = q_1^{d_1} \dots q_k^{d_k} \cdot p^d$, where $0 \leqslant d_i \leqslant c_i$, $1 \leqslant i \leqslant k$, and $0 \leqslant d \leqslant c$. If $i \in \{1, \dots k\}$ and $d_i \neq 0$, then let Q_i be a non-abelian metacyclic group of order $q_i^{2c_i+h}$ having a maximal normal cyclic subgroup of order $q_i^{c_i+h}$ and a non-abelian metacyclic quotient $\bar{Q}_i$ of order $q_i^{2d_i+h}$ having a maximal normal cyclic subgroup $\bar{N}_i$ of order $q_i^{d_i+h}$, where $h = 2$ if $q_i = 2$ and $h = 1$ if q_i is odd. Such Q_i exist by I.1.9. If $i \in \{1, \dots, k\}$ and $d_i = 0$, then set Q_i to be an extraspecial group of order $q_i^{2c_i+1}$ and $\overline{Q}_i$ to be a cyclic group of order q_i. Then $\overline{Q}_i \in \mathcal{Q}(Q_i)$. Let L be a splitting field for Q_i and $\overline{Q}_i$ of characteristic p. By I.4.7, I.4.8 and I.2.16, there is a faithful irreducible submodule V_i for Q_i and $\overline{V}_i$ for $\overline{Q}_i$ over L of dimension $q_i^{c_i}$ and $q_i^{d_i}$ repectively. Notice that as q_i is $\mathfrak{R}$-potent (by IV.3 (c)) and therefore $1 \in \mathfrak{R}_{q_i}$, then $Q_i \in \mathfrak{F}_a(\mathfrak{R})$, $1 \leq i \leq k$. Let $E = E(p, q)$, where q is some $\mathfrak{R}$-potent prime different from p. If $a \neq 0$, then q must exist, since any group G having a faithful and irreducible module over a field of characteristic p cannot be a p-group. Let V be a faithful irreducible E-module over L. If L is chosen so that it is also a splitting field for E, then $\dim_L V = p$ (as in IV.3 (c)). The group E belongs to $\mathfrak{F}_a(\mathfrak{R})$ by IV.3 (a). Set

$$H = Q_1 \times \dots \times Q_k \times E \times \overset{c}{\dots} \times E \in \mathfrak{F}_a(\mathfrak{R}).$$

By I.4.12, the L H-module $V_1 \otimes_L \dots \otimes_L V_k \otimes_L V \otimes_L \overset{c}{\dots} \otimes_L V$ is absolutely irreducible of dimension $q_1^{c_1} \dots q_k^{c_k} p^c = a \in \mathfrak{R}_p$. By I.4.13, this module is faithful for H, and so $H \in \Gamma(p)$. Now let

$$\overline{H} = \overline{Q}_1 \times \dots \times \overline{Q}_k \times E \times \overset{d}{\dots} \times E \in \mathcal{Q}(H) \subseteq \Gamma(p).$$

Then again by I.4.12, the L H-module $\overline{V}_1 \otimes_L \dots \otimes_L \overline{V}_k \otimes_L V \otimes_L \overset{d}{\dots} \otimes_L V$ is absolutely irreducible and of dimension $q_1^{d_1} \dots q_k^{d_k} p^d = b$. Since $\overline{H} \in \Gamma(p)$, we deduce that $b \in \mathfrak{R}_p$.

(b) Since $\mathfrak{R}$ is minimal, there exist groups G and H in $\Gamma(p)$ such that if L is a splitting field of characteristic p for $G \times H$ and its subgroups, then there are faithful irreducible modules V and W for G and H respectively over L of dimensions

a and b, respectively. By I.4.12, $V \otimes_L W$ is an absolutely irreducible L (G x H)-module of dimension a.b. Since G x H $\in$ f (p), we conclude that a.b$\in \mathcal{R}_p$.
q.e.d.

<u>IV.5 Corollary</u> The sets $\mathcal{R}_p$ are completely determined by the primes belonging to them.

<u>IV.6 Corollary</u> $\mathfrak{F}_a(\mathcal{R})$ is S_n-closed.

<u>Proof</u> This follows easily from IV.4 (b) and Clifford's Theorem, using I.3.1.
q.e.d.

Corollary IV.6 is something of a relief after the corresponding result in Chapter III (III.1.2). Indeed, we notice that $\mathcal{R}$ has some of the nicer aspects of a full ranking function $\mathcal{S}$ such that $\mathfrak{F}(\mathcal{S})$ is saturated. In fact, our ranking function $\mathcal{R}$ is very much nicer that such $\mathcal{S}$, and the next result even implies that conjecture III.2.11 holds for $\mathcal{R}$.

<u>IV.7 Proposition</u> (c.f. property R F 5) If p and q are $\mathcal{R}$-potent primes and $p \neq q \in \mathcal{R}_p$ then $\mathcal{R}_p$ contains all $\mathcal{R}$-potent primes.

<u>Proof</u> Let t be an $\mathcal{R}$-potent prime. If t = q then $t \in \mathcal{R}_p$. Suppose, therefore, that $t \neq q$. Let $T \cong Z_t$, $Q \cong Z_q$, and let V be a non-trivial irreducible T-module over GF (q) of dimension m say. Let L be a splitting field for $H = Q \wr_T [V] \cdot T$ of characteristic p which is a Galois extension of K = GF (p). Let U be a non-trivial irreducible L Q-module – i. e. $\dim_L U = 1$. Setting [illegible] $= U\mathfrak{U}_Q VT$ and using the notation of II.1.17 we see that U^* may be considered as a faithful irreducible H-module over L of

dimension q^m. Since $q \in \mathfrak{R}_p$, we have $q^m \in \mathfrak{R}_p$. By IV.3 (a) we have $V.T \in \mathfrak{F}_a(\mathfrak{R})$ and $T \in f(q)$. Since every q-chief factor of H induces a group of automorphisms in H which is isomorphic to a quotient of T, it follows that $H \in \mathfrak{F}_a(\mathfrak{R})$. Let Y be an irreducible $K\,H$-submodule of $(U^*)_K$. Then $r_a(Y) = \dim_L U^* \in \mathfrak{R}_p$ when Y is considered as a chief factor of $[Y] \cdot H$. Hence every chief factor of $Y.H$ is $\mathfrak{R}_a$-admissible and $H \in f(p)$. In particular, $V.T \in f(p)$. Since an absolutely irreducible faithful $V.T$-module has dimension t, we have $t \in \mathfrak{R}_p$.

<u>IV.8 Proposition</u> (c. f. Property R F 6) Let $p \in \mathbb{P}$ with $p \in \mathfrak{R}_p$. Then $\mathfrak{R}_q \subseteq \mathfrak{R}_p$ for all primes q.

<u>Proof</u> Clearly we may concentrate on an $\mathfrak{R}$-potent prime $q \neq p$. By IV.5, in order to prove that $\mathfrak{R}_q \subseteq \mathfrak{R}_p$ it is enough to show that if $r \in \mathbb{P}$ and $r \in \mathfrak{R}_q$, then $r \in \mathfrak{R}_p$. Let $T \cong Z_r$ and let $E = E(p, q)$. By IV.3 (b), the group E belongs to $\mathfrak{F}_a(\mathfrak{R})$. Let $W = E \wr T$. If $B \leq E$ then we denote the image of B in the base group of W by B^*. By II.1.17, if $N = F(E)$, then N^* is the unique minimal normal subgroup of W and W is, in fact, primitive. Also, $r(N^*) = r \cdot r(N)$. Let H be a complement of N in E. Then H^*T is a complement of N^* in W. By I.4.3, we have $r_a(N^*) \,\big|\, |H^*T : H^*| = r$ and so, because H^*T is not abelian, it follows that $r_a(N^*) = r \in \mathfrak{R}_q$. Since $T \in f(p)$ by IV.3 (a), the p-chief factors of W must be $\mathfrak{R}_a$-admissible. Hence every chief factor of W is $\mathfrak{R}_a$-admissible, and so $W \in \mathfrak{F}_a(\mathfrak{R})$. By II.1.17 (c) and I.2.16 there is a faithful and irreducible module V of dimension p^a over a splitting field L for W and each subgroup of W of characteristic p. Since $p \in \mathfrak{R}_p$ it follows

that $p^a \in \mathfrak{R}_p$. Thus if U is an irreducible GF(p) W-submodule of $V_{GF(p)}$, then $[U] \cdot W \in \mathfrak{F}_a(\mathfrak{R})$ and so $W \in f(p)$. Now, because W is primitive, there is an integer n such that the group $[N^* \times \underset{n}{\ldots} \times N^*] \cdot H^* T$ is faithfully and irreducibly represented on a module of dimension $|H^* T|$ over L. Since $[N^* \times \underset{n}{\ldots} \times N^*] \cdot H^* T \in R_0(W) \subseteq f(p)$, we must have $|H^* T| \in \mathfrak{R}_p$. By IV.4 we therefore conclude that $r \in \mathfrak{R}_p$.

q.e.d.

<u>IV.[illegible] Corollary</u> Let $\pi \subseteq \mathbb{P}$ be the set of $\mathfrak{R}$-potent primes. Let π_1, π_2 and π_3 be subsets of π such that

$$\pi_1 = \{p \in \pi : \mathfrak{R}_p = \{1\}\},$$

$$\pi_2 = \{p \in \pi : \mathfrak{R}_p = \langle \pi \rangle\},$$ where $\langle \pi \rangle$ is the set of all products of primes from π, and

$$\pi_3 = \{p \in \pi : \mathfrak{R}_p = \{p^i : i \geq 0\}\}.$$

Then

(<u>a</u>) $\pi = \pi_1 \cup \pi_2 \cup \pi_3$;

(<u>b</u>) $\pi_2 \cap \pi_3 = \emptyset$;

(<u>c</u>) if $\pi_2 \neq \emptyset$ then $\pi = \pi_1 \cup \pi_2$; and

(<u>d</u>) if $\pi_3 \neq \emptyset$ then $\pi = \pi_1 \cup \pi_3$ and $|\pi_3| = 1$.

<u>Proof</u> (<u>a</u>) Suppose that $p \in \pi \setminus (\pi_1 \cup \pi_3)$. Then there is an $\mathfrak{R}$-potent prime q with $p \neq q \in \mathfrak{R}_p$. Propositions IV.7 and IV.4 then force $\langle \pi \rangle \subseteq \mathfrak{R}_p$. It now follows from IV.3 (i) that $\mathfrak{R}_p = \langle \pi \rangle$, and hence $p \in \pi_2$.

(<u>b</u>) If $p \in \pi_2 \cap \pi_3$, then we have $\{p^i : i \geq 0\} = \langle \pi \rangle$ – that is $\pi = \{p\}$ and $\mathfrak{F}_a(\mathfrak{R}) = \mathfrak{G}_p$. However, all chief factors of a p-group have rank 1, contradicting the minimality of $\mathfrak{R}$. Hence $\pi_2 \cap \pi_3 = \emptyset$.

(c) Suppose that $\pi_2 \neq \emptyset$, and let $q \in \pi_2$. If $p \in \pi_3$ then by IV.9 we have $\mathfrak{R}_q = \langle \pi \rangle \subseteq \mathfrak{R}_p$, implying that $p = q \in \pi_2 \cap \pi_3 \neq \emptyset$, against (b). Therefore, $\pi_3 = \emptyset$.

(d) If $\pi_3 \neq \emptyset$, then it is immediate from (c) that $\pi_2 = \emptyset$. It is, furthermore, immediate from IV.8 that $|\pi_3| = 1$.

q.e.d.

Corollary IV.9 already gives us a clear impression of the nature of $\mathfrak{F}_a(\mathfrak{R})$.

<u>IV.10 Definition</u> The class $\mathfrak{U}^{+}$ of <u>absolutely supersoluble</u> groups is the class of soluble groups in which every chief factor has absolute rank 1. If π is a set of primes then $\mathfrak{U}^{+}(\pi)$ will denote the class of <u>absolutely π-supersoluble</u> groups - that is, those soluble groups in which every π-chief factor has rank 1.

<u>IV.11 Lemma</u> Let π be the characteristic of $\mathfrak{F}_a(\mathfrak{R})$ and let π_1, π_2 and π_3 be a partition of π as described in IV.9. Then $\mathfrak{F}_a(\mathfrak{R})$ is described by one of the following:

(a) if $\pi = \pi_1 \cup \pi_2$ then $\mathfrak{F}_a(\mathfrak{R}) = \mathfrak{U}^{+}(\pi_1) \cap \mathfrak{S}_\pi$, the class of π_1-supersoluble π-groups.

(b) if $\pi = \pi_1 \cup \pi_3$ and $\pi_3 = \{p\}$, then $\mathfrak{F}_a(\mathfrak{R})$ has the following full and local definition:

$$g(q) = \mathfrak{A}_\pi \text{ if } q \in \pi_1$$

$$g(p) = \mathfrak{A}_\pi \mathfrak{A}_p.$$

<u>Proof</u> (a) This is clear.

(b) Let $\mathfrak{G}$ be the formation defined by g.

(1) <u>$\mathfrak{G} \subseteq \mathfrak{F}_a(\mathfrak{R})$</u>: Let $G \in \mathfrak{G}$ and let M / N be a q-chief factor of G. If $q \in \pi_1$, then $A_G(M / N) \in g(q)$ and so $A_G(M / N)$ is cyclic. In particular, $r_a(M / N) = 1 \in \mathfrak{R}_q$. If $q = p$, then

$A_G(M/N) \in \mathfrak{A}_\pi\mathfrak{A}_p$, and it follows from I.4.3 that $r_a(M/N)$ is a power of p. Hence $G \in \mathfrak{F}_a(\mathfrak{R})$.

(2) $\mathfrak{F}_a(\mathfrak{R}) \subseteq \mathfrak{G}$: Let $G \in \mathfrak{F}_a(\mathfrak{R})$, and let M/N be a q-chief factor of G. If $q \neq p$, then we must have that $r_a(M/N) = 1$, and hence $A_G(M/N)$ is a cyclic π-group. In particular, $A_G(M/N) \in \mathfrak{g}(q)$. Suppose, then, that $q = p$. We wish to show that $A_G(M/N) \in \mathfrak{A}_\pi\mathfrak{A}_p$ in order to complete the proof. Let $\bar{G} = A_G(M/N)$. If $r_a(M/N) = 1$ then G is cyclic and the result is clear. We may therefore assume that $r_a(M/N) = p^b$, some $b \geq 1$. By I.4.3 therefore, $p^b \mid |\bar{G}|$. Let L be a splitting field for $\bar{G}$ and all its subgroups, and let V be an irreducible $L\bar{G}$-submodule of $(M/N)^L$. Since V is faithful for $\bar{G}$ we have $F(\bar{G}) \subseteq O_{p'}(\bar{G})$. On the other hand, $V|_{O_{p'}(\bar{G})}$ is a direct sum of irreducible $L O_{p'}(\bar{G})$-modules of dimension a power of p. If W is an irreducible submodule of $V|_{O_{p'}(\bar{G})}$ then by [26], VI, 8.1 we deduce that

$O_{p'}(\bar{G}) \in R_0\mathfrak{A}_{\pi_1} = \mathfrak{A}_{\pi_1}$. Further, since now $O_{p'}(\bar{G})$ is nilpotent,

$F(\bar{G}) = O_{p'}(\bar{G})$. Since $F(\bar{G})$ is abelian, it must be that $F(\bar{G})$ is the stabilizer in $\bar{G}$ of W, and so $V \cong W^{\bar{G}}$. Because W is an irreducible $L F(\bar{G})$-module, we have $\dim_L W = 1$, and as $\dim_L V = p^b$, we therefore see that $|\bar{G} : F(\bar{G})| = p^b$. We now conclude that the p-chief factors of $\bar{G}$ are central in $\bar{G}$, and from II.1.11 and [26], VI, 8.1 that the p'-chief factors of $\bar{G}$ induce automorphism groups in $\bar{G}$ that are cyclic p-groups. Thus $\bar{G} / \bigcap C_{\bar{G}}(X/Y)$ is an abelian p-group, where the intersection is taken over all chief factors X/Y of $\bar{G}$. By II.1.11 we therefore have $\bar{G} / F(\bar{G}) \in \mathfrak{A}_2$, completing the proof.

q.e.d.

After a technical lemma, we are able to state and prove the theorems characterising all absolutly ranked saturated formations.

<u>IV.12 Lemma</u> Let $\pi \subseteq \mathbb{P}$ and let $\mathfrak{G}$ be a formation with char $\mathfrak{G} \subseteq \pi$. Then $\mathfrak{N}_\pi \mathfrak{G}$ is a saturated formation.

<u>Proof</u> By II.3.5, $\mathfrak{N}_\pi \mathfrak{G}$ is a formation. Let G be a group and suppose that $G / \Phi(G) \in \mathfrak{N}_\pi \mathfrak{G}$. Let $H \leq G$ be such that $\Phi(G) \leq H$ and $H / \Phi(G) = (G / \Phi(G))^{\mathfrak{G}}$. Since $H / \Phi(G) \in \mathfrak{N}_\pi$ we have $H / \Phi(G) \leq F(G / \Phi(G))$. By II.1.1 (c), it now follows that $H \leq F(G)$. Since $G / \Phi(G)$ is a π-group, then G is also a π-group (by [26], III, 3.8). Because $G / H \in \mathfrak{G}$ and $\mathfrak{N}_\pi$ is S-closed, we deduce that $G \in \mathfrak{N}_\pi \mathfrak{G}$.

q.e.d.

<u>IV.13 Theorem</u> Let $\mathfrak{R}$ be a ranking function. The following statements are equivalent in pairs:

(<u>a</u>) $\mathfrak{F}_a(\mathfrak{R})$ is saturated.

(<u>b</u>) There are disjoint sets π_1 and π_2 of primes, with $|\pi_1 \cup \pi_2| = 1$ only if $\pi_2 = \emptyset$, such that char $\mathfrak{F}_a(\mathfrak{R}) = \pi_1 \cup \pi_2$, $\mathfrak{R}_p = \{1\}$ if $p \in \pi_1$, and precisely one of the following statements holds:

(i) $\pi_2 = \emptyset$.

(ii) $\pi_2 \neq \emptyset$ and $\mathfrak{R}_p = \langle \pi_1 \cup \pi_2 \rangle$ for all $p \in \pi_2$.

(iii) $|\pi_2| = 1$ and $\mathfrak{R}_p = \{p^i : i \geq 0\}$ for $\pi_2 = \{p\}$.

(<u>c</u>) There are disjoint sets π_1 and π_2 of primes with $|\pi_1 \cup \pi_2| = 1$ only if $\pi_2 = \emptyset$, such that either

(i) $\mathfrak{F}_a\,\mathfrak{R} = \mathfrak{S}_{\pi_2}\mathfrak{N}_{\pi_1\cup\pi_2}\mathfrak{A}_{\pi_1\cup\pi_2}$, or

(ii) $|\pi_2| = 1$ and $\mathfrak{F}_a\,\mathfrak{R}$ is locally defined by g with

$g(q) = \mathfrak{A}_{\pi_1\cup\pi_2}$ for all $q \in \pi_1$, and

$g(p) = \mathfrak{A}_{\pi_1\cup\pi_2}\mathfrak{A}_{\pi_2}$, where $\pi_2 = \{p\}$.

Proof (a)$\Rightarrow$(b) Most of this has been done in IV.2. Suppose that $\pi_1 = \emptyset$ and $|\pi_2| = 1$. Then in either case (ii) or (iii) we have $\mathfrak{R}_2 = \{p^i : i \geqslant 0\}$, where $\pi_2 = \{p\}$. But now $\mathfrak{F}_a\,\mathfrak{R} \subseteq \mathfrak{S}_p$. This contradicts the minimality of $\mathfrak{R}$, since every chief factor of a p-group has rank 1.

(b)$\Rightarrow$(c) Suppose firstly that either (b) (i) or (b) (ii) holds and let $G \in \mathfrak{S}_{\pi_2}\mathfrak{N}_{\pi_1\cup\pi_2}\mathfrak{A}_{\pi_1\cup\pi_2}$. Let M / N be a chief factor of G. Since G is a $\pi_1\cup\pi_2$-group, it follows from I.4.3 and the definition of π_2 that all π_2-chief factors of G are $\mathfrak{R}_a$-admissible. We may therefore assume that M / N is a π_1-chief factor. Since the $\mathfrak{N}_{\pi_1\cup\pi_2}\mathfrak{A}_{\pi_1\cup\pi_2}$-residual of G will centralise M / N, we may assume without loss of generality that this residual is trivial. Let $K = G^{\mathfrak{A}_{\pi_1\cup\pi_2}}$. By II.1.15, there is a chief factor M_0 / N_0 of a chief series of G running through K which is G-isomorphic to M / N. If M_0 / N_0 lies above K then it is central in G and therefore certainly $\mathfrak{R}_a$-admissible. Suppose that M_0 / N_0 lies below K. Since K is a nilpotent normal subgroup of G, it is contained in $F(G)$. Then by II.1.11, K centralises M_0 / N_0. Hence

$A_G(M_0 / N_0) \in Q(G / K) \subseteq \mathfrak{A}_{\pi_1\cup\pi_2}$. Thus $A_G(M_0 / N_0)$

is cyclic and $r_a(M_0 / N_0) = r_a(M / N) = 1$. Hence

$\mathfrak{S}_{\pi_2}\mathfrak{N}_{\pi_1 \cup \pi_2}\mathfrak{A}_{\pi_1 \cup \pi_2} \subseteq \mathfrak{J}_3\,\mathfrak{R}$.

Now let $\mathfrak{G} = \mathfrak{N}_{\pi_1 \cup \pi_2}\mathfrak{A}_{\pi_1 \cup \pi_2}$. Let $G \in \mathfrak{J}_3(\mathfrak{R})$ and set $K = G^{\mathfrak{G}}$.

We claim that $K = K^{\mathfrak{S}_{\pi_1}}$. Suppose that $K^{\mathfrak{S}_{\pi_1}} < K$. Since $K^{\mathfrak{S}_{\pi_1}}$ char $K \trianglelefteq G$

we may find a normal subgroup J of G such that $K^{\mathfrak{S}_{\pi_1}} \leq J < K$ and K / J is a chief factor of G. Since $K / J = (G / J)^{\mathfrak{G}}$, it follows from [4], Theorem 5.15 that K / J is a complemented chief factor of G. Let L be a complement of K / J in G. Then

$$[K / J] \cdot A_G(K / J) \cong G / \mathrm{core}_G(L).$$

On the other hand, K / J is a π_1-chief factor of G and so must induce a cyclic group of automorphisms in G, as $r_a(K / J) = 1$. Hence $L / \mathrm{core}_G(L) \in \mathfrak{A}_{\pi_1 \cup \pi_2}$ and therefore

$G / \mathrm{core}_G(L) \in \mathfrak{N}_{\pi_1}\mathfrak{A}_{\pi_1 \cup \pi_2} \subseteq \mathfrak{G}$. But the definition of K now

forces $K \leq \mathrm{core}_G(L) \leq L$, contradicting the fact that L

complements the chief factor K / J. Therefore K is $\mathfrak{S}_{\pi_1}$-perfect,

as claimed. Next set $M = K^{\mathfrak{S}_{\pi_2}}$ and suppose that $M > 1$. Then

there is a chief factor M / N of G. Since $N \trianglelefteq K$ we deduce that

M / N is a π_1-chief factor. As therefore $r_a(M / N) = 1$, we

must have $A_G(M / N) \in \mathfrak{A}_{\pi_1 \cup \pi_2}$. Hence $K \leq C_G(M / N)$. We may choose a chief factor M / M_1 of K with $N \leq M_1 < M$ which, by the Schur-Zassenhaus Theorem, is complemented in K. Then $M / M_1 \cong [M / M_1] \cdot A_K(M / M_1) \in \mathfrak{Q}(K)$ since $K \leq C_G(M / N) \cap K = C_K(M / N) \leq C_K(M / M_1)$. This however, contradicts the fact that K is $\mathfrak{S}_{\pi_1}$-perfect.

Therefore $1 = K^{\mathfrak{S}_{\pi_2}}$ and hence

$$G \in \mathfrak{S}_{\pi_2}\mathfrak{G} = \mathfrak{S}_{\pi_2}\mathfrak{N}_{\pi_1 \cup \pi_2}\mathfrak{A}_{\pi_1 \cup \pi_2}.$$

If (b) (iii) holds then $\mathfrak{F}_a(\mathfrak{R})$ has the stated local definition by IV.11.

<u>(c) $\Rightarrow$ (a)</u>: By IV.12 the formation $\mathfrak{N}_{\pi_1 \cup \pi_2}\mathfrak{A}_{\pi_1 \cup \pi_2}$ is a saturated formation. Then $\mathfrak{S}_{\pi_2}\mathfrak{N}_{\pi_1 \cup \pi_2}\mathfrak{A}_{\pi_1 \cup \pi_2}$ is saturated by [19], VI, Theorems 3, 6. The class described in (c) is saturated by II.3.10 (a).

q.e.d.

<u>IV.14 Theorem</u> If π_1 and π_2 are disjoint sets of primes with $|\pi_1 \cup \pi_2| = 1$ only if $\pi_2 = \emptyset$, then

(a) there is a ranking function $\mathfrak{R}$ such that

$$\mathfrak{F}_a(\mathfrak{R}) = \mathfrak{S}_{\pi_2}\mathfrak{N}_{\pi_1 \cup \pi_2}\mathfrak{A}_{\pi_1 \cup \pi_2}, \text{ and}$$

(b) if $|\pi_2| = 1$, then there is a ranking function $\mathfrak{R}$ such that $\mathfrak{F}_a(\mathfrak{R})$ is locally defined by

$$g(q) = \mathfrak{A}_{\pi_1 \cup \pi_2} \text{ for } q \in \pi_1$$

$$g(p) = \mathfrak{A}_{\pi_1 \cup \{p\}}\mathfrak{A}_{\{p\}} \text{ for } \pi_2 = \{p\}.$$

<u>Proof</u> (a) Set $\mathfrak{R}_q = \{1\}$ for $q \in \pi_1$ and $\mathfrak{R}_p = \langle \pi_1 \cup \pi_2 \rangle$ for $p \in \pi_2$.

Now apply IV.13.

(b) Set $\mathfrak{R}_q = \{1\}$ for $q \in \pi_1$ and $\mathfrak{R}_p = \{p^i : i \geq 0\}$ for $\pi_2 = \{p\}$.

Now apply IV.13.

q.e.d.

Theorems IV.13 and IV.14 have completely determined the absolutely ranked saturated formations.

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